Disagreement, Financial Markets, and the Real Economy

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Abstract

I study consumption, asset prices, and portfolios in a production economy with two agents who disagree regarding firm productivity. The economy is characterized by continual overconsumption by individuals, and periodic "consumption booms" at the aggregate level. An optimistic investor believes stocks offer high excess returns, whereas a pessimist perceives low or negative excess returns. Each investor is able to construct a portfolio that seems to offer higher returns than he could achieve without the presence of his "misinformed" counterpart. As a result, each believes that his portfolio can support a higher level of consumption. When the wealth distribution tilts toward the optimist, aggregate consumption rises to a level not seen in either agent's homogeneous economy: a consumption boom. In a multi-sector version of the economy, controversy regarding one small firm is sufficient to cause significant movements in the price of a larger firm.

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1 Introduction

This paper links disagreement among investors in financial markets to the real economy. Investors disagree regarding firm productivity. Consequently they assign different expected returns to a firm's stock. This disagreement spills into the money market, as different expected stock returns support different risk-free rates. When prices are determined in equilibrium, the relative returns on investments are altered from those the agents would encounter in homogeneous economies of their respective types. This has two main consequences for the real economy. One is that the allocation of capital among firms may be distorted. Another is that the trade-off between consumption and saving is altered, possibly leading to overconsumption or overinvestment. These effects are seen in the portfolios of individual agents, and at an aggregate level.

To examine these issues, I present two calibrations of the canonical linear AK production economy extended to allow for two agent types. The agents are identical except in their perceptions of firm productivity. The first calibration focuses on questions related to consumption and saving, and assumes a single firm type (or sector). One agent is optimistic, expecting high firm productivity, whereas the other is pessimistic. A key result is that both agents "overconsume" in the heterogeneous economy relative to how they would consume in their respective homogeneous economies. The effects are strongest when an agent holds a small fraction of the economy's total wealth. When the pessimist is a small player in a market dominated by optimists, he consumes a fraction of his wealth almost 50 percent higher than he would consume in an economy dominated by pessimists. Likewise the optimist facing a market dominated by pessimists consumes over 40 percent more than in a purely optimistic economy. Overconsumption is more moderate when wealth is more evenly distributed, but agents always consume more in heterogeneous economies than in their respective homogeneous economies.

Agent perceptions of securities markets explain these results. The money market is particularly important. In a standard representative agent economy, the equilibrium risk-free rate is such that no trade occurs in the money market. With different agent perceptions of expected stock returns, the equilibrium risk-free rate falls between the lower rate that would obtain in the homogeneous pessimist's economy and the higher one of the homogeneous optimist's economy. Hence the risk-free rate offered to the pessimist in the heterogeneous market seems attractive relative to the low expected stock returns he perceives. Consequently he lends in the money market. The optimist his happy to be his counterparty, as to him the risk-free rate appears low:

in his estimation the stock offers high excess returns, so he takes a levered long position in the stock. The essential point is that each agent perceives his portfolio as obtaining higher (risk-adjusted) returns in the mixed economy than he could obtain in his homogeneous economy. In a sense, given he has the same amount of capital to invest in either case, each agent feels richer when he has a "misinformed" counterparty to trade with. Under the assumption that agents have an elasticity of inter-temporal substitution less than 1, this leads them to consume a greater fraction of their wealths in the heterogeneous economy. [1]

The interplay between wealth distribution, prices, and individual agent consumption leads to nontrivial dynamics in aggregate consumption that run counter to the intuition from homogeneous economies. When the firm is productive, the stock return is high, and wealth shifts toward the optimist. In isolation, the optimist would consume more of his wealth than the pessimist (approximately 27 % more in my calibration), so one might expect greater optimist wealth to result in higher aggregate consumption. However in the mixed economy a wealthier optimist implies a higher risk-free rate. This raises the pessimist's consumption (as he invests in the money market), but lowers the optimist's consumption (the leveraged stock position is less attractive). Whether high stock returns increase or decrease aggregate consumption depends upon how wealth is initially distributed. Aggregate consumption achieves its maximum when the optimist controls roughly 85 % of aggregate wealth. In this economic state both the optimist and the pessimist consume more than would be consumed in a homogeneous optimist economy. Such occurrences are called "consumption booms."

To study the effects of disagreement upon capital allocation between firms, I introduce a two-firm parameterization of the model where agents disagree moderately about the productivity of one firm. Parameters are chosen such that the controversial firm is relatively small, with approximately 15 % market share in the initial period. However its market share may grow to over 25 % if stock returns favor the optimist, or sink to as little as 5 % if they favor the pessimist. Despite these shifts, aggregate consumption and saving behavior remains almost unchanged. Basic accounting implies that "spillover" from the controversy regarding the small firm significantly influences the price of the large firm. The altered capital allocation affects the bond market through changes in the variance of aggregate output. There is little diversification under the pessimist, which leads to greater variation in aggregate output and a lower risk-free rate. The optimist invests

[1] In the calibrated examples that follow I assume constant relative risk aversion $\gamma = 2$, which corresponds to $EIS = 1/2$. As one might expect, an $EIS > 1$ leads to "underconsumption" in heterogeneous economies, and the knife-edge case of log utility implies that each agent consumes the same as he would in his homogeneous economy.

more in the small firm, which reduces variance in output and supports a higher risk-free rate. Therefore a shift in wealth toward the optimist produces two effects: greater investment in the small firm (and hence higher total investment in stocks), and a higher risk-free rate (which depresses investment in stocks). The result is a shift in allocation across sectors, but little change in total capital allocated to production.

Related Literature

Numerous papers investigate the impact of investor disagreement upon asset prices. Recent examples include Banerjee and Kremer [2010], David [2008], Dumas et al. [2009], Gallmeyer and Hollifield [2008], Li [2007], and Yan [2008]. Each of these papers incorporates interesting belief dynamics, frictions, or additional forms of heterogeneity for increased realism, or to address empirical asset pricing phenomena such as trading volume, price volatility, or the equity premium. However they restrict analysis to endowment economies in the style of Lucas [1978], in which investor disagreement impacts prices and portfolios but leaves aggregate consumption and the path of economic growth unchanged. The purpose of my paper is to address precisely those issues entwined with macroeconomic quantities not addressed in this extensive literature. My results have implications for the extension of heterogeneous beliefs asset pricing results to the production setting. For example, the price volatility produced by my model is far smaller than that which would obtain in an endowment economy with similar disagreement regarding the aggregate dividend process. In line with Jermann [1998], this suggests that some form of capital adjustment friction is required to extend disagreement-based explanations of price volatility to a production setting.

In contrast to the endowment setting, theoretical studies of production economies with investor disagreement are few. Detemple and Murthy [1994] is closest to this paper. They also investigate a frictionless production economy with investor disagreement, obtaining closed-form results for prices and portfolios in a continuous-time framework with one productive sector. However they restrict preferences to log utility, which leaves consumption unaffected by beliefs. More recently Branch and McGough [2011] investigate the effects of belief heterogeneity on the internal propagation of real business cycle models. Their work focuses the consumption and output dynamics induced by TFP shocks, as opposed to connections between markets, portfolios and consumption.

The paper is organized as follows. Section 2 describes the model. Section 3 solves the planner's problem, and characterizes prices and portfolios of the corresponding competitive equilibrium. Section 4 presents the single-firm version of the model. Section 5 considers a two-firm parameterization. Section 6 concludes.

2 The Model

I construct a discrete-time, discrete-state stochastic optimal growth model in the style of Brock and Mirman [1972], but extended to two agents and M firms. However I abstract from labor. There is a single good, which is both consumable and used as capital in production. The next subsections describe firms, households, and markets.

2.1 Firms

There are M competitive firms responsible for production. Each firm $i \in M$ has access to a simple linear technology that produces output according to $F_t^i = \tilde{A}_t^i K_{t-1}^i$, where F_t^i is output produced at the beginning of period t, K_{t-1} is capital input selected at the end of the previous period $t-1$, and $\tilde{A}_t^i$ is the stochastic productivity of technology i, which is observed at the beginning of period t. The input K_t^i is fully depreciated in the production process, and there are no capital adjustment costs or other frictions to concern the firm. Define the vector $\tilde{A}_t = [\tilde{A}_t^1, \tilde{A}_t^2, \ldots, \tilde{A}_t^M]$. I assume $\tilde{A}_t$ is discrete and i.i.d. each period with outcomes $A_t \in \mathbb{A} \subset \mathbb{R}^M$, $\forall t$, and denote the number of states $N = |\mathbb{A}| < \infty$. Note that although I assume $\tilde{A}$ is i.i.d. over time, I allow productivity to be correlated across firms.

At time $t = 0$, each firm i raises capital by issuing a single, infinitely divisible share of stock. It maximizes the value of its stock by announcing a plan for optimal dividends,

$$\begin{aligned} p_0^{s,i} &= \max_{\{d_t^i\}_{t=1}^{\infty}} E_0 \left[\sum_{t=1}^{\infty} m_t d_t^i \right] \\ \text{s.t. } d_t^i &= F_t^i - K_t^i, \\ F_t^i &= \tilde{A}_t^i K_{t-1}^i, \end{aligned} \tag{1}$$

where $K_0^i = p_0^{s,i}$ is the value of capital raised in the IPO, and m_t is a stochastic discount factor, which I discuss in section 3.

2.2 Households

The economy is populated by an optimist and a pessimist, $j \in \{O, P\}$, who live forever. The agents have identical CRRA preferences over consumption of the numeraire,

$$u(c^j) = \frac{(c^j)^{1-\gamma}}{1-\gamma},\ u(c^j) = \log(c^j)\ \text{ if }\ \gamma = 1. \tag{2}$$

In the first period, agent O is endowed with a fraction $\theta_O \in [0, 1]$ of the initial stock of the good, and agent P receives the remainder, $\theta_P = 1 - \theta_O$. Through investment of his endowment, each agent maximizes his expected utility of lifetime consumption,

$$\begin{aligned} &\max_{\{c_t^j\}_{t=0}^{\infty}} E_0^j\left[\sum_0^{\infty} \beta^t u(c_t^j)\right] \\ &s.t.\ E_0^j\left[\sum_{t=0}^{\infty} m_t^j c_t^j\right] \leq \theta_j f_0, \end{aligned} \tag{3}$$

where $\beta \in (0, 1)$ is a constant reflecting impatience, and E_t^j denotes expectation with respect to agent j's beliefs given the information set at time t. Although identical in most respects, the two agents disagree about the probability distribution of the firm productivities $\tilde{A}$. That is, although they agree on the state space (their measures are equivalent), they assess state probabilities according to $P_O[\tilde{A} = A]$, $A \in \mathbb{A}$ and $P_P[\tilde{A} = A]$, respectively. Therefore the two agents will not generally agree on the optimal way to invest a given amount of wealth. In fact, the technology actually behaves according to a third distribution, that of the econometrician, $P_E[\tilde{A} = A]$. Although it is reasonable to assume that rational agents would eventually learn the true distribution of $\tilde{A}$, O and P are irrational insofar as their beliefs are dogmatic.[2]

For analytical convenience I introduce a representative agent who is a composite of O and P. In each period, the representative agent solves

$$\bar{u}(c, \lambda) = \max_{c^O} u(c^O) + \lambda u(c - c^O) \tag{4}$$

where c is current aggregate consumption, λ is a weighting factor reflecting the current relative wealth of the two agents, and $c^P = c - c^O$. λ evolves stochastically over time according to the law of motion

$$\lambda_t = \tilde{z}_t \lambda_{t-1}, \tag{5}$$

[2]Even if agents are fully rational learners, different beliefs may persist for long periods given sufficiently different priors. This is particularly true if the productivity process were to be made more realistic. See for example [Yan, 2008, p. 1946].

where $\tilde{z}_t$ is a discrete random variable that is i.i.d. across periods (and hence, as with $\tilde{A}$, I will usually omit the time subscript). $\tilde{z}$ represents disagreement between the agents regarding the probabilities of technological outcomes in the next period. It is a "one-period change of measure" from the beliefs of O to those of P, i.e., $\tilde{z} = \frac{\lambda_{t+1}}{\lambda_t} = \frac{P_P[\tilde{A}=A]}{P_O[\tilde{A}=A]}$, $\forall A \in \mathbb{A}$, $\forall t$. λ_0 is a constant chosen such that each agent's budget constraint is satisfied, an issue that I address in a later section. [3]

The above formulation of the representative agent's utility function and the accompanying law of motion for λ implicitly assume that we will optimize expected utility under the optimist's measure. Henceforth and without loss of generality, I take expectations with respect to beliefs of the optimist O unless otherwise specified. [4]

Taking aggregate consumption and the weighting factor as given, the solution to Equation (4) is

$$c^O(c, \lambda) = \frac{c}{1 + \lambda^{1/\gamma}}, \tag{6}$$

from which it follows that the utility of the representative agent is

$$\bar{u}(c, \lambda) = \frac{c^{1-\gamma}}{1-\gamma} \left(1 + \lambda^{1/\gamma}\right)^{\gamma}. \tag{7}$$

Note that $\bar{u}(c, \lambda)$ is homogeneous of degree $1 - \gamma$ in c.

2.3 Markets

Markets are dynamically complete, with as many linearly independent assets as states N, and trade is frictionless. In particular, I assume that there is a one period bond (or equivalently a money market), which pays one unit of the good regardless of technological outcome, and one share of stock in each firm. With the exception of the stocks, which are each in net supply one, all assets are assumed to be in zero net supply.

[3]For a practical discussion of change of measure in a discrete setting, see for example [Shreve, 2004, p. 61]. For additional discussion of the representative investor in a heterogeneous beliefs economy, see Basak [2005].

[4]The model could equivalently be solved under the pessimist's measure by placing the stochastic weighting factor λ on the optimist's utility $u(c^O)$, and defining $\tilde{z} = \frac{P_O[\tilde{A}=A]}{P_P[\tilde{A}=A]}$ instead. The resulting equilibrium capital allocation would be the same.

3 Equilibrium

Equilibrium is determined by two key state variables realized at the start of each period: the stochastic weighting factor λ_t, and aggregate output $f_t = \sum_{i=1}^{M} F_t^i$. The fact that we need only keep track of aggregate output and not sectoral allocations is due to the assumption of frictionless capital adjustment. I solve for equilibrium in two main steps: first I solve the planner's problem, then I construct the corresponding competitive equilibrium.

To solve the planner's problem, I determine optimal aggregate consumption c_t and capital allocation $K_t = [K_t^1, K_t^2, ..., K_t^M]'$ taking the initial weighting factor λ_0 and aggregate stock of the good f_0 as given. Subsequently I construct a stochastic discount factor m_t. This is used to derive the wealth w_t^j of each agent as a function of λ_t and f_t, which is finally used to determine λ_0 consistent with the initial wealth allocations given by θ_O, θ_P and f_0.

With the solution to the planner's problem in hand, the prices and portfolios forming the competitive equilibrium derive from the stochastic discount factor m_t and the wealth functions w_t^j. I discuss stock prices and the risk-free rate, and explain the solution procedure for portfolios.

All proofs are in the appendix.

3.1 Aggregate Saving and Capital Allocation

I select K_t as the choice variable, denoting aggregate saving $k_t = \sum_{i=1}^{M} K_t^i$ and aggregate consumption $c_t = f_t - k_t$. The objective is to solve for an optimal policy function $K_t = K(f_t, \lambda_t)$ that gives capital allocation as a function of the state variables. Formulated as a sequence problem, the value function is defined as

$$\begin{aligned} \bar{v}(f_0, \lambda_0) = \max_{\{K_t\}_{t=0}^{\infty}} E_0 &\left[\sum_{t=0}^{\infty} \beta^t \bar{u}(f_t - k_t, \lambda_t) \right] \\ s.t.\ \lambda_t &= \tilde{z}_t \lambda_{t-1}, \\ f_t &= \tilde{A}_t K_{t-1}. \end{aligned} \tag{8}$$

Typically we also desire nonnegative and bounded capital allocations $K_t^i \in [0, 1)$, $k_t \in (0, 1)$. I do not enforce this formally, but it results naturally from the form of the utility function and judicious selection of parameters.

Due to the infinite horizon setting, the solution can be given in terms of the state variables f and λ without explicit reference to time. Under technical conditions the value function can be formulated recursively as

$$\bar{v}(f,\lambda) = \max_{K} \bar{u}(f-k,\lambda) + \beta E[\bar{v}(\tilde{A}K, \tilde{z}\lambda)], \tag{9}$$

The right hand side has first order conditions

$$\left(\frac{1+\lambda^{1/\gamma}}{f-k}\right)^{\gamma} = \beta E[\bar{v}'(\tilde{A}K, \lambda\tilde{z})\tilde{A}^{i}],\ i \in \{1,\dots,M\}. \tag{10}$$

The envelope theorem implies $\bar{v}'(f,\lambda) = \bar{u}'(f-k(f,\lambda),\lambda)$. In combination with first order conditions, this yields the Euler equations,

$$\left(\frac{1+\lambda^{1/\gamma}}{f-k(f,\lambda)}\right)^{\gamma} = \beta E\left[\left(\frac{1+(\tilde{z}\lambda)^{1/\gamma}}{\tilde{A}K(f,\lambda) - k(\tilde{A}K(f,\lambda), \tilde{z}\lambda)}\right)^{\gamma} \tilde{A}^{i}\right],\ i \in \{1,\dots,M\}. \tag{11}$$

The functional form of the solution is characterized as follows.

Proposition 1. *Any function $K(f,\lambda)$ satisfying Equation (11) is homogeneous of degree one in f, and has the form $K(f,\lambda) = fB(\lambda)$, where $B : \mathbb{R}_{+} \to \mathbb{R}^{M}$ satisfies*

$$\left(\frac{1+\lambda^{1/\gamma}}{1-b(\lambda)}\right)^{\gamma} = \beta E\left[\left(\frac{1+(\tilde{z}\lambda)^{1/\gamma}}{1-(\tilde{A}B(\lambda))b(\tilde{z}\lambda)}\right)^{\gamma} \tilde{A}^{i}\right],\ i \in \{1,\dots,M\} \tag{12}$$

for $b(\lambda) = \sum_{i=1}^{M} B^{i}(\lambda)$.

Unfortunately a closed form solution to Equation (12) seems unlikely. However an accurate numerical approximation to the solution can be computed without difficulty. Details are available upon request.

3.2 State Prices

The representative agent values consumption in a given time and state according to a stochastic discount factor m_t reflecting his marginal utility of consumption in that time and state. Specifically, if t is the current period, then

$$m_{t+1} = \beta\frac{\bar{u}'(c_{t+1},\lambda_{t+1})}{\bar{u}'(c_t,\lambda_t)} = \beta\frac{u'(c_{t+1}^{O})}{u'(c_t^{O})} = \beta\tilde{z}\frac{u'(c_{t+1}^{P})}{u'(c_t^{P})}. \tag{13}$$

The equivalence of the representative agent's SDF and those of P and O is implied by the envelope theorem, and is easily verified algebraically. By way of example, let a_{t+1} be an asset that offers an arbitrary stochastic payoff in the next period only, and recall that $\tilde{z}$ is a change of measure from beliefs of P to those of O. Therefore the price of a_{t+1} is

$$p_a = E_t^O\left[\beta\frac{u'(c_{t+1}^O)}{u'(c_t^O)}a_{t+1}\right] = E_t^O\left[\beta\tilde{z}\frac{u'(c_{t+1}^P)}{u'(c_t^P)}a_{t+1}\right] = E_t^P\left[\beta\frac{u'(c_{t+1}^P)}{u'(c_t^P)}a_{t+1}\right]. \tag{14}$$

Whether agent P values a_{t+1} under his beliefs and according to his marginal utility of consumption, or O does the same using his beliefs and marginal utility, the value of a_{t+1} is the same. So O and P disagree about state probabilities, but agree upon state prices. An important implication is that, when the firm maximizes the value of its shares, the choice of measure and SDF is irrelevant. Finally, homogeneity of $\bar{u}$ in c and of K in f imply that we can express

$$m_{t+1} = m(\lambda_t) = \beta\left(\frac{1+\lambda_t^{1/\gamma}}{1-b(\lambda_t)}\right)^{-\gamma}\left(\frac{1+(\tilde{z}\lambda_t)^{1/\gamma}}{(1-b(\tilde{z}\lambda_t))(\tilde{A}B(\lambda))}\right)^{\gamma}, \tag{15}$$

i.e. the SDF does not depend upon the current period level of aggregate output.

3.3 Wealth and Budget Constraints

I define an agent's wealth as the discounted value of his contingent claims for present and future consumption. Here I develop an expression for the wealth of the optimist, $w_O(f,\lambda)$, as a function of the aggregate capital stock and weighting factor. With $w_O(f,\lambda)$ in hand it is easy to solve for λ_0 such that each agent's budget constraint is satisfied, and also to determine their optimal portfolios. O's wealth (calling the current period $t = 0$) is defined as

$$w_O = E_0\left[\sum_{t=0}^{\infty} m_t c_t^O\right]. \tag{16}$$

Using the laws of motion for f and λ, wealth can be expressed recursively as

$$\begin{aligned} w_O(f,\lambda) &= c_O(f,\lambda) + \beta E\left[\frac{c_O(\tilde{A}B(\lambda)f,\tilde{z}\lambda)^{-\gamma}}{c_O(f,\lambda)^{-\gamma}}w_O(\tilde{A}B(\lambda)f,\tilde{z}\lambda)\right] \\ &= \frac{f(1-b(\lambda))}{1+\lambda^{1/\gamma}} + \beta E\left[\left(\frac{\tilde{A}B(\lambda)(1-b(\tilde{z}\lambda))}{1+(\tilde{z}\lambda)^{1/\gamma}}\right)^{-\gamma}\left(\frac{1-b(\lambda)}{1+\lambda^{1/\gamma}}\right)^{\gamma} w_O(\tilde{A}B(\lambda)f,\tilde{z}\lambda)\right] \end{aligned} \tag{17}$$

A function of the form $w_O(f,\lambda) = D(\lambda)f$ will satisfy the equation, where $D(\lambda)$ satisfies

$$D(\lambda) = \frac{1-b(\lambda)}{1+\lambda^{1/\gamma}} + \beta E\left[\left(\frac{1-b(\tilde{z}\lambda)}{1+(\tilde{z}\lambda)^{1/\gamma}}\right)^{-\gamma}\left(\frac{1-b(\lambda)}{1+\lambda^{1/\gamma}}\right)^{\gamma} D(\tilde{z}\lambda)(\tilde{A}B(\lambda))^{1-\gamma}\right]. \tag{18}$$

The function $D(\lambda)$ gives the optimist's fraction of aggregate wealth. With one additional assumption it is possible to establish the existence of a unique solution to Equation (18).

Assumption 1. *Let* $\hat{\beta} = \sup_{\lambda} \beta E\left[\left(\frac{1-b(\tilde{z}\lambda)}{1+(\tilde{z}\lambda)^{1/\gamma}}\right)^{-\gamma}\left(\frac{1-b(\lambda)}{1+\lambda^{1/\gamma}}\right)^{\gamma}(\tilde{A}B(\lambda))^{1-\gamma}\right]$*. Assume model parameters s.t.* $\hat{\beta} < 1$.

Although satisfaction of Assumption 1 cannot generally be verified analytically, I verify it computationally for parameters used in numerical results later in the paper.

Proposition 2. *Define a space of continuous, bounded functions* $\mathcal{D}(\Lambda)$, $\Lambda \equiv \mathbb{R}_+$, $g \in \mathcal{D}(\Lambda)$ *s.t.* $g : \Lambda \to [0,1]$*, with the sup norm* $\|g\| = \sup_{\lambda\in\Lambda} |g(\lambda)|$*. Let* $\mathcal{T}_D$ *be the mapping given by Equation (18),*

$$[\mathcal{T}_D g](\lambda) = \frac{1-b(\lambda)}{1+\lambda^{1/\gamma}} + \beta E\left[\left(\frac{1-b(\tilde{z}\lambda)}{1+(\tilde{z}\lambda)^{1/\gamma}}\right)^{-\gamma}\left(\frac{1-b(\lambda)}{1+\lambda^{1/\gamma}}\right)^{\gamma} g(\tilde{z}\lambda)(\tilde{A}B(\lambda))^{1-\gamma}\right]. \tag{19}$$

Under Assumption 1, there is a unique solution $D(\lambda) \in \mathcal{D}(\Lambda)$ *to Equation (18), and* $\forall g \in \mathcal{D}(\Lambda)$, $\lim_{N\to\infty}[\mathcal{T}_D^N g](\lambda) \to D(\lambda)$*.*

The optimist's wealth w_O is not simple to characterize analytically for general parameters. The exception is log utility ($\gamma = 1$), in which case $w_O(f,\lambda) = \frac{f}{1+\lambda}$, i.e., the agent's share of wealth is identical to his share of consumption. As with K, I compute w_O numerically for general parameters satisfying Assumption 1.

In order to complete the equilibrium solution the initial weighting factor λ_0 must be chosen such that each agent's budget constraint is satisfied. That is, λ_0 is chosen to solve

$$w_O(f,\lambda_0) = \theta_O f_0. \tag{20}$$

Since the pessimist holds any endowment and consumption claims not held by the optimist, satisfaction of O's budget constraint also implies satisfaction of P's.

3.4 Asset Prices

Using the optimal capital allocation that solves the planners problem, I derive prices supporting a competitive equilibrium. From the stochastic discount factor, the price of the one-period bond is

$$\begin{aligned} p_b(\lambda) &= E[m(\lambda)] \\ &= \beta\left(\frac{1+\lambda^{1/\gamma}}{1-b(\lambda)}\right)^{-\gamma} E\left[\left(\frac{1+(\tilde{z}\lambda)^{1/\gamma}}{(1-b(\tilde{z}\lambda))(\tilde{A}B(\lambda))}\right)^{\gamma}\right], \end{aligned} \tag{21}$$

and the gross risk-free rate is simply the reciprocal of the bond price,

$$r(\lambda) + 1 = \frac{1}{p_b(\lambda)}. \tag{22}$$

Following logic equivalent to that of [Cox et al., 1985, p. 382], the ex-dividend price of the stock must be equal to the capital held by the firm, i.e. for firm i,

$$p_s^i(f, \lambda) = K^i(f, \lambda). \tag{23}$$

The basic argument behind this result relies on the assumptions of free entry and frictionless capital adjustment. If $p_s^i > K^i$, then the firm's owners could profitably sell their shares and invest K^i in a new firm employing the same technology (and hence capable of generating the same dividend stream as the original firm). If $p_s^i < K^i$, then the firm can liquidate its capital and pay K^i to its owners as a dividend, who could then invest K^i in a new firm employing the same technology (where by nature of the IPO the value of the firm would be $p_s' = K^i$). The fact that the firm's owners desire K^i to be invested in technology i follows by the fact that they are (as a group) the representative agent, and it is precisely this allocation that supports the optimal aggregate dividend stream, $c(f, \lambda) = \sum_{i=1}^{M} d^i = f - \sum_{i=1}^{M} K^i(f, \lambda)$.

The fact that price is equal to capital is significant for stock returns. Gross returns are

$$\frac{d_{t+1}^i + p_{t+1}^i}{p_t^i} = \frac{F_{t+1}^i}{K_t^i} = \tilde{A}_{t+1}^i, \tag{24}$$

that is, the gross returns on a firm's stock are equal to the productivity of the firm's technology. It follows that gross returns are i.i.d., and that disagreement regarding technology processes could equally be characterized as disagreement regarding the distribution of stock returns.

3.5 Portfolios

In equilibrium, each agent follows a portfolio rule that implements his optimal wealth process, which in turn supports his optimal consumption plan. Having solved for the wealth of the optimist as a function of the state variables, I reverse engineer his optimal portfolio rule to satisfy this relationship.

Let $R(f, \lambda)$ be an NxN matrix of gross asset payoffs, such that $R_{j,i}(f, \lambda)$ is the next-period payoff of asset i in state j given the current state is described by (f, λ).[5] Let $\phi(f, \lambda)$ be an $Nx1$ vector implementing the portfolio rule, and finally $\Omega(f, \lambda)$ an $Nx1$ vector describing target wealth outcomes in each possible next-

[5] Of course it is possible to have any number of assets $\geq N$ s.t. the rank of R is N, but I assume for simplicity that the portfolio rule uses the minimal number of assets.

period state. Then

$$
\begin{aligned}
R(f,\lambda)\phi(f,\lambda) &= \Omega(f,\lambda) \\
\Rightarrow \phi(f,\lambda) &= R^{-1}(f,\lambda)\,\Omega(f,\lambda)
\end{aligned}
\tag{25}
$$

Given the optimist O's portfolio, that of the pessimist is recoverable by market clearing.

In particular, consider the case $N = 2$, $M = 1$ with only the stock and the bond as assets.[6] Then the optimal portfolio satisfies

$$
\phi(f,\lambda) = \begin{bmatrix} A_1 k(f,\lambda) & 1 \\ A_2 k(f,\lambda) & 1 \end{bmatrix}^{-1} \begin{bmatrix} w_O(A_1 k(f,\lambda), Z_1\lambda) \\ w_O(A_2 k(f,\lambda), Z_2\lambda) \end{bmatrix}
\tag{26}
$$

4 An Economy with One Firm

To provide a simple example that offers some intuition regarding the behavior of the economy, I continue with the one firm, two state formulation introduced in the previous section. This has the advantage of restricting portfolios to the two assets of primary interest, the stock and the one-period bond, as additional assets needn't be introduced in order to complete markets.

Parameter	Value	Description
$\mathbb{A}$	{H=1.14, L=1.02}	Production technology states
p_P	{0.4, 0.6}	Beliefs, agent P (pessimist)
p_O	{0.6, 0.4}	Beliefs, agent O (optimist)
p_E	{0.5, 0.5}	Beliefs, econometrician
β	0.99	Impatience
f_0	1	Initial stock of the good
θ_O	0.5	Initial wealth share, agent O
γ	2	Relative risk aversion

Table 1: Baseline parameter values.

I calibrate the model's parameters to match some aspects of the US economy. There are three probability

[6]Analogous to $A = \{A_1, A_2\}$, let $Z = \{Z_1, Z_2\}$ be the state space of $\tilde{z}$.

measures: those of the pessimist P, the optimist O, and the econometrician E, where the latter is the measure by which the economy actually evolves. As E plays no active role in the economy (he holds no assets), his beliefs have not been involved in the derivation of equilibrium, but they determine the dynamics of the state variables. They also provide a point of reference. The econometrician knows that each state in $\mathbb{A}$ is equally likely. The optimist believes the high state is more likely than the low state by 0.1, whereas the pessimist believes the low state is more likely, also by 0.1. This amounts to a substantial level of disagreement between the agents, but their errors in beliefs are symmetric about the econometrician's. By selecting equal initial wealth shares $\theta_O = \theta_P = 1/2$ I make a wealth-weighted average of the population's beliefs equal to the econometrician's, i.e., on average the agents are correct.

Table 1 gives the full set of parameters. Although the agents are dogmatic in their divergent beliefs, they agree on the levels of productivity corresponding to the two technology states. I choose the high (H) and low (L) states of A such that, in a homogeneous beliefs economy with agents believing as the econometrician, the mean and variance of output growth match the sample mean and variance of annual US real GDP growth from 1930 through 2009, as recorded by the BEA. [7] Mean growth is around 3.41%, with standard deviation 5.07%. To match these, the technology has mean return $\mu_A = 7.74\%$ and standard deviation $\sigma_A = 7.52\%$. The agents have identical, moderate risk aversion $\gamma = 2$. The level of risk aversion was chosen to highlight non-monotonicity in savings that is less apparent (but still present) with higher risk aversion.

Obviously a more precise parameter estimation is possible, taking the levels of disagreement and risk aversion as free parameters and attempting to match additional empirical moments. My objective is a simple calibration that highlights the basic implications of the model. Alternative parameters were explored in previous drafts of this paper; the results presented here are representative of those the model produces. One less innocuous assumption is $\gamma > 1$ and consequently $EIS < 1$, which determines whether the main result of disagreement is overconsumption or overinvestment. Selecting $\gamma < 1$ would, roughly speaking, flip the consumption figures about the horizontal axis. Since it is more common to assume $\gamma > 1$ I do so for purposes of example.

[7] See http://www.bea.gov/national/index.htm. I use the data for chained 2005 dollars.

4.1 Evolution of the Optimist's Consumption Share

After developing the model it is clear that the state variable of interest is λ, the stochastic weighting factor, so I present its dynamics now. The level of aggregate output f acts simply as a linear scaling factor on consumption, wealth, and stock prices, and does not directly impact state-prices or the interest rate. I discuss dynamics of f - which are influenced by those of λ - in a later section.

Productivity (A_1)	Econometrician's Probability	z_1	λ_1	ω_1
L	1/2	3/2	3/2	0.45
H	1/2	2/3	2/3	0.55

Table 2: Period $t = 1$ values of the pareto weight λ_1 and optimist's consumption share ω_1 given realized productivity under the baseline parameters under the simplifying assumption $\lambda_0 = 1$ ($\omega_0 = 0.5$). Low productivity increases the weight on the pessimist's utility and decreases the optimist's consumption share. The opposite occurs given high productivity. Under the baseline parameters, agent disagreement is symmetric about the econometrician's beliefs, which assign each state a 50% probability. It follows that λ is as likely to increase as decrease, and the magnitude of an increase is the reciprocal magnitude of the decrease. Both agents will survive indefinitely in this economy.

There is only one exogenous process driving this model: the productivity of the firm, $\tilde{A}$. The realization of $\tilde{A}$ will determine that of change of measure $\tilde{z}$, which will in turn drive λ and most variables of economic interest. For example, consider the optimist's share of aggregate consumption, $\omega \in [0, 1]$,

$$\omega(\lambda) = \frac{1}{1 + \lambda^{1/\gamma}}. \tag{27}$$

If productivity is low (L), which the pessimist finds more likely than the optimist, then the realization z^L of $\tilde{z}$ is greater than one. This increases λ, the Pareto weight on the pessimist's utility, which implies a decrease in the optimist's consumption share ω. The *value* of $\tilde{z}$ in each state (high (H) or low (L) productivity) is determined by the relative beliefs of the optimist and the pessimist, but the *probability* that high or low productivity occurs is determined under the econometrician's beliefs, which reflect the true behavior of the production technology. Assuming for simplicity that $\lambda_0 = 1$, Table 2 provides a concrete example under the baseline parameters of what occurs in the next period $t = 1$, given realized productivity.

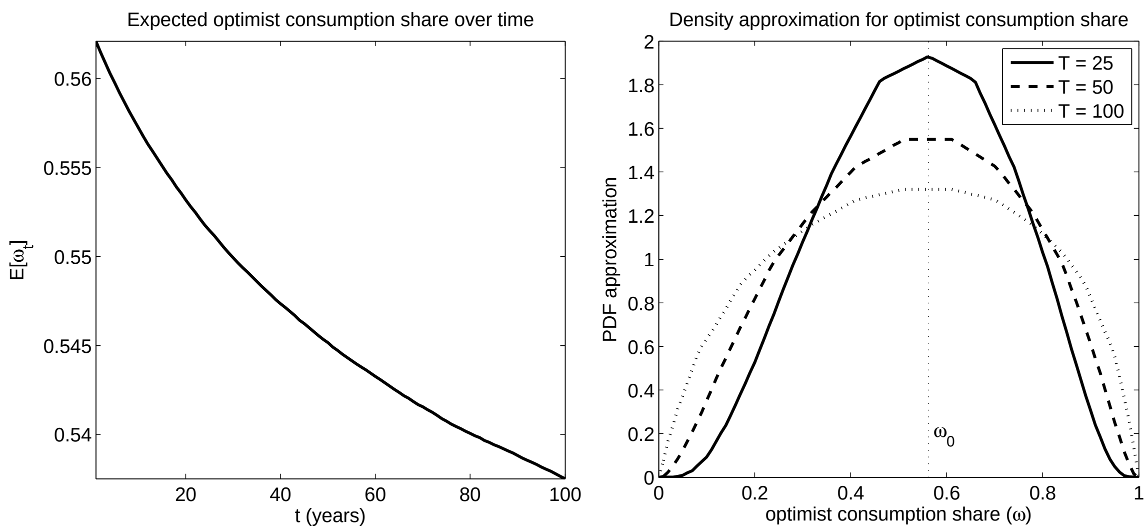

Figure 1: Expectation and density of the optimist's consumption share (ω_t) over time. Initial value $\omega_0 \approx 0.56$ is chosen to satisfy individual budget constraints. At longer horizons ω_t is more likely to be far from its initial position. Although ω_t has a discrete distribution, for purposes of visualization the right panel shows a continuous approximation to the probability mass function.

I characterize economic behavior using ω as the state variable. The optimist's share of aggregate consumption, ω, is bounded between 0 and 1, and has a more natural interpretation than λ, which is unbounded above. There is also a monotonic mapping between the two variables. An alternative would be to use the optimist's wealth share $D(\lambda)$ as the state variable, but it is not available in closed form, which makes D a more complicated and less intuitive option.

Figure 1 shows the evolution of the optimist's consumption over time, in expectation and in distribution. The baseline parameters assign each agent equal wealth in the initial period. In contrast to the simplifying assumption in Table 2, this usually *does not* imply an equal consumption share $\omega_0 = 0.5$, or equivalently $\lambda_0 = 1$. To satisfy individual budget constraints under baseline parameters a value of $\omega_0 \approx 0.56$ is required, i.e., the optimist initially consumes more of his wealth than the pessimist. This is reflected in the plots. Consumption share is initially tilted slightly towards the optimist, and this is still visible in the distribution of ω_{25}, at the 25 year horizon. Over time the impact of ω_0 washes out, and $E[\omega_t]$ goes to 0.5. As the economy evolves, successive productivity realizations allow ω_t to move further away from its initial value, as shown in the right panel. At long horizons there is a significant probability that the economy is dominated by the pessimist ($\omega_t \approx 0$) or by the optimist ($\omega_t \approx 1$). Although ω_t can come arbitrarily close to 0 or 1, it never reaches the absorbing states; both agents will survive indefinitely. In fact, as time increases without bound, each value in the domain of ω is reached with probability 1. Therefore it is reasonable to consider economic behavior over the entire domain of ω, while keeping in mind that states near ω_0 are more likely at short horizons. The next few sections explain the economic impact of the shifting consumption share. For a more detailed discussion of the asymptotic behavior of ω (and correspondingly λ), see the appendix.

4.2 Consumption

Most prior studies of disagreement take place in endowment economies, where aggregate consumption is exogenous. If the optimist and pessimist were situated in an endowment economy with disagreement about the growth of c_t, then ω_t would retain its interpretation, with the optimist consuming $c_t\omega_t$ and the pessimist the remainder $c_t(1-\omega_t)$.[8] If c_t experienced a positive exogenous shock, ω_{t+1} would increase versus ω_t, and optimist would consume more, both as a fraction of the total and in absolute terms. Likewise the pessimist's consumption must decrease in the same way, such that total consumption equals exogenous c_t; the consumption "decisions" of the two agents are entwined. The obvious distinction in the production setting is that each agent determines - in some respects independently of the other - how much of his wealth he will consume. The sum of these consumption choices determines aggregate consumption endogenously. The optimist's consumption share ω_t remains sufficient as a state variable (with f_t), but it is no longer sufficient to describe the optimist's consumption *decision*.

The right panel of Figure 2 shows how much of his wealth each agent chooses to consume, as a function of state variable ω. At the leftmost extreme of the plot, ω approaches 0, the optimist is relatively impoverished, and the pessimist dominates the economy. The pessimist behaves as if the economy is a homogeneous one populated by pessimists.[9] He consumes little, as he expects slow economic growth. By contrast the comparatively poor optimist consumes a large portion of his wealth, roughly 80% more than the pessimist, and roughly 40 % more than he would consume if he were the only agent type in the economy. These results have nothing to do with each agent's absolute wealth; they obtain whether aggregate output f is high or low.

[8] So the state variables in the endowment economy would be c_t and ω_t, with c_t replacing f_t.

[9] In a homogeneous economy the agent consumes a constant fraction of his wealth, and his wealth is equal to aggregate output f.

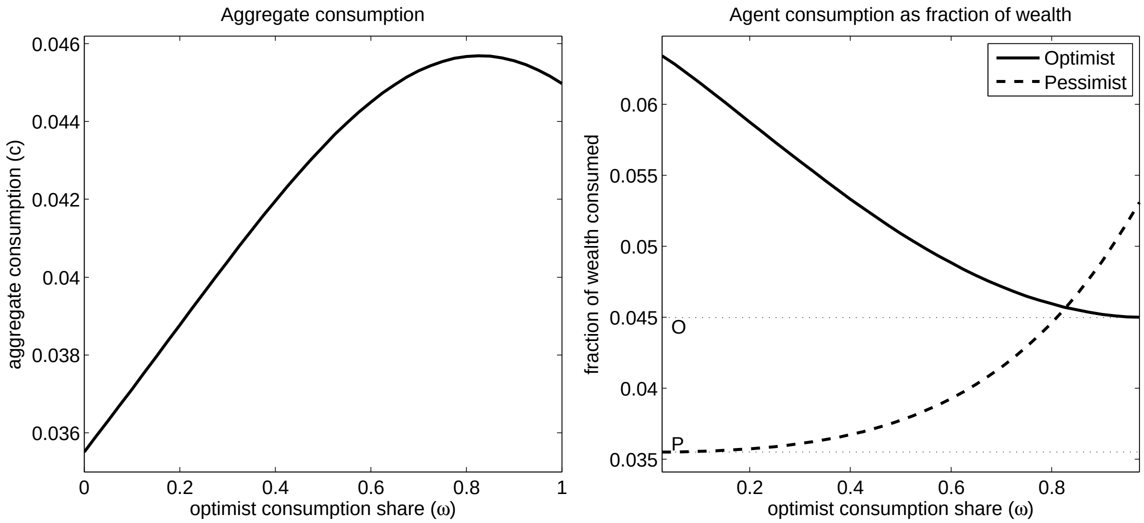

Figure 2: Aggregate consumption and individual consumption as a fraction of wealth.

As we move rightwards on the horizontal axis, the optimist's consumption share increases, as does his fraction of total wealth. But the fraction of his wealth *that he consumes* decreases sharply, whereas the pessimist consumes more of his wealth as his economic influence wanes. As ω approaches 1, the optimist follows his homogeneous economy policy, whereas the pessimist's consumption is sharply elevated, by roughly 50 % relative to wealth.

These results are remarkable in three respects. First, both agents always over-consume in the heterogeneous economy, although the magnitude of overconsumption declines to nothing as an agent becomes dominant. Second, when the pessimist has little economic influence ($\omega \to 1$), he consumes a substantially higher fraction of his wealth than the optimist does. This is in sharp contrast to the homogeneous economy analysis, in which, ceteris paribus, the pessimist consumes significantly less than the optimist. Finally, for a wide interval on ω, both agents over-consume substantially versus their homogeneous levels, and in the rightmost fifth of the plot both agents consume more than the optimist's homogeneous consumption level.

These facts are apparent in results for aggregate consumption, in the left panel of Figure 2. Consistent with individual consumption results, the leftmost extreme of the plot, where $\omega = 0$, corresponds to the pessimist's homogeneous economy consumption rule. Likewise the rightmost extreme with $\omega = 1$ matches the homogeneous optimist economy. But rather than displaying a weighted average characteristic, aggregate consumption is always higher than the weighted average of the homogeneous economy levels, because both agents are overconsuming. When the average investor is quite optimistic, around $\omega \approx 0.8$, a consumption boom occurs; both optimistic and pessimistic investors over-consume sufficiently that aggregate consumption is driven to levels beyond the range of either the pessimist's or the optimist's homogeneous economy level. In contrast, Detemple and Murthy [1994] find that aggregate consumption is a weighted average of the homogeneous economy levels, and individual agents do not alter their consumption rules from their homogeneous economy plans. These results are due to the assumption of log utility, which I relax.

Casting back to the dynamic properties of the model illustrated by Figure 1, the plausible range of values for ω_t becomes quite broad after a century has passed. It is likely that boom and bust cycles will occur over the course of decades, even though productivity is i.i.d.

4.3 Asset Prices, Portfolios, and Perceptions

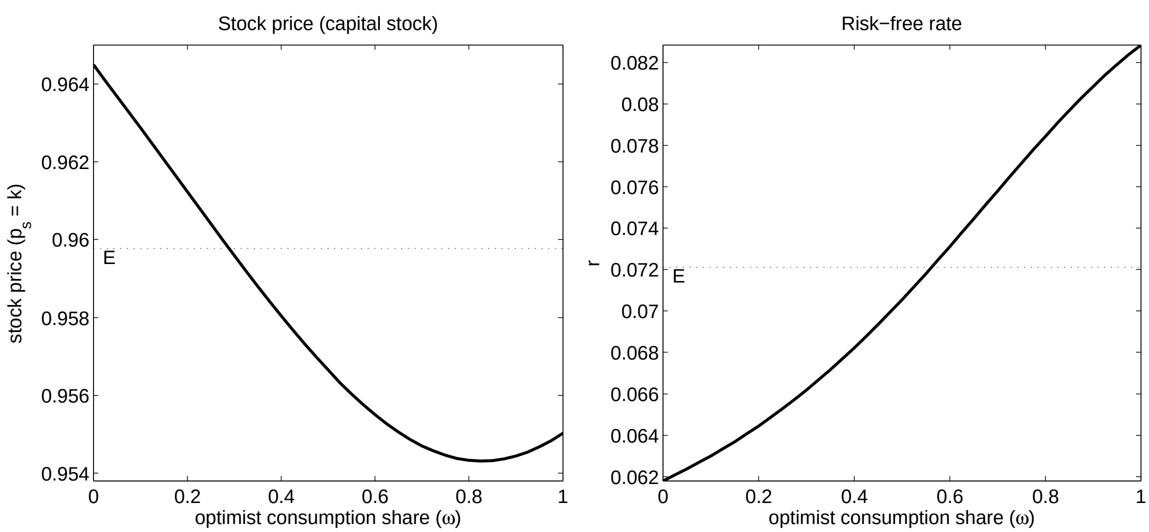

Figure 3: Stock price and risk-free rate. The stock price assumes $f = 1$.

The consumption behavior of the previous section becomes more intuitive once competitive equilibrium prices and portfolios are examined. The risk-free rate is the main determinant of portfolios, whereas the price of the stock is important in relation to economic growth. Each of these is shown in Figure 3. The risk-free rate r, in the right panel, exhibits a weighted average characteristic. Intuitively the rate is higher when the average investor is optimistic (at the right of the plot) than when he is pessimistic (at the left), and the difference of the two extremes - over 200 basis points - is rather large. As a point of reference, the risk-free rate that would occur under the econometrician's beliefs (which are the midway between the optimist and the pessimist) is found in the center of the range, at the dotted line marked 'E'.

The stock price (identical to the firm's capital stock) is found in the left panel, assuming for simplicity that aggregate output $f = 1$. Equivalently the plot may be interpreted as the stock price relative to aggregate output (price/GDP). In the single sector economy the shape of the stock price curve is simply a mirror image of aggregate consumption, as $p_s = f - c$. Although aggregate consumption varies substantially in response to shifts in consumption share, by roughly 25 % from peak to trough, this corresponds to relatively minor stock price fluctuations, of only 1% between extremes. This result is very different from what obtains in a simple endowment economy with heterogeneous beliefs, where comparable levels of disagreement would generate price fluctuations orders of magnitude larger.[10] As discussed in Jermann [1998] and elsewhere, the frictionless linear production model with CRRA utility does a poor job of explaining asset prices. Although difference of opinion induces time-variation in a model that would otherwise have a constant risk-free rate and price to GDP ratio, the risk-free rate is too high, and price fluctuations are far too small. The model's performance in this regard would likely be improved by introducing capital adjustment costs or other frictions, and calibrating with a view toward asset pricing.

From the perspective of investors in this model, price volatility is irrelevant: it affects how returns are decomposed into price appreciation and dividend yields, but returns remain i.i.d. What is essential to the investor's decision making is his perception of expected stock returns relative to the risk-free rate. In the top left panel, Figure 4 shows each agent's perception of excess returns of the stock. The econometrician's reading of the true excess return falls midway between the two agents. Although investors agree upon the risk-free rate, the optimist perceives much higher excess returns than the pessimist. When the pessimist is dominant ($\omega \approx 0$), the risk-free rate is low. However the pessimist also expects low stock returns, so his perceived excess return approaches his homogeneous economy level. Accordingly he holds the stock and does not lend significantly. (Portfolio weights on bonds and stocks are shown in the left and right panels of Figure 4, respectively.) However the optimist facing the same risk-free rate perceives very high excess returns. His wealth is trivial, but he borrows heavily relative to his wealth to take a levered long position in the stock. Perceptions of the Sharpe Ratio, in the top right panel, also correspond closely to excess returns,

[10]For examples of price volatility due to heterogeneous beliefs in an endowment economy, see for example Li [2007] and Dumas et al. [2009].

and support each agent's portfolio strategy.

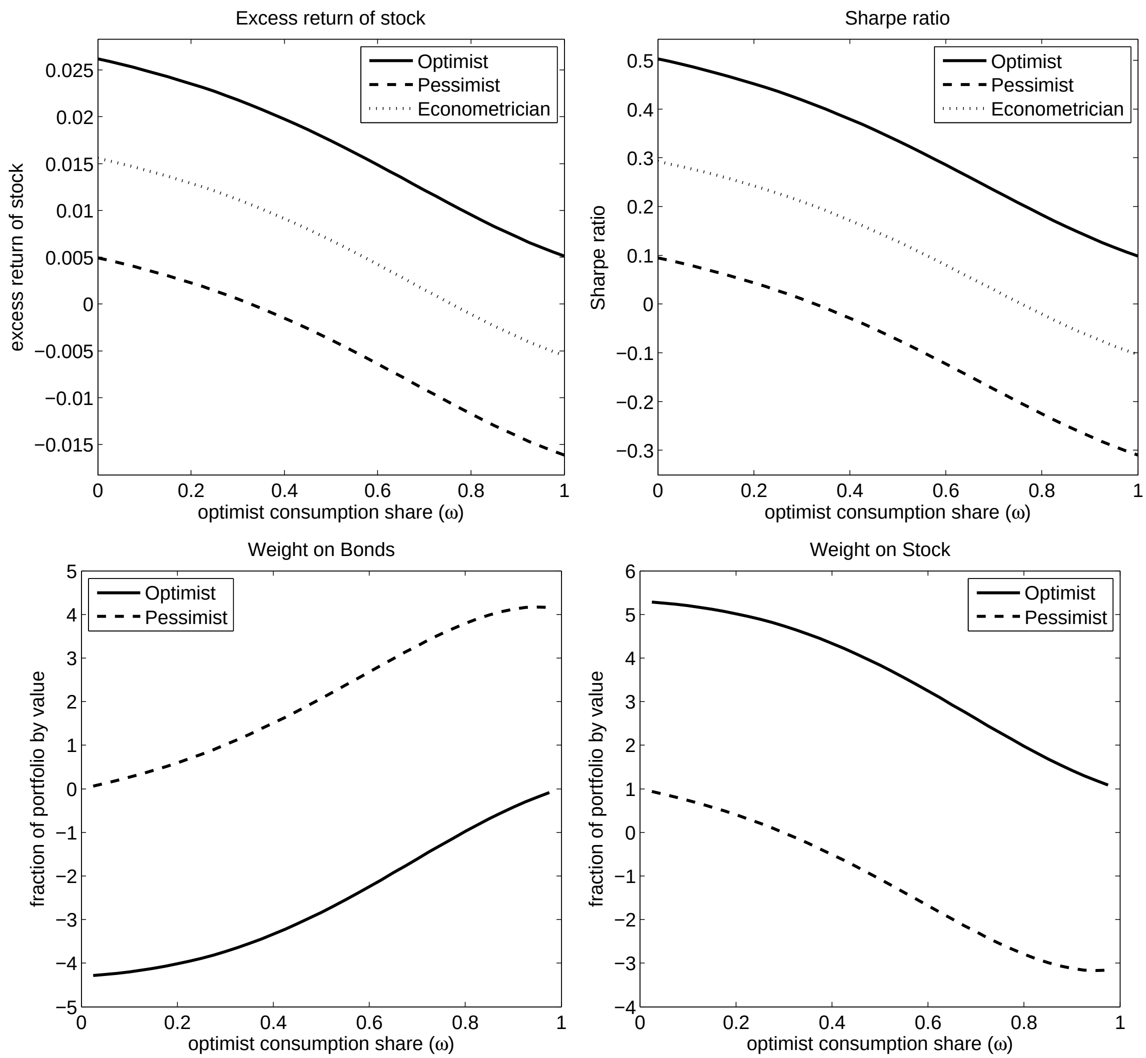

Figure 4: Excess returns and the Sharpe ratio as perceived by each agent, and his resulting investment decisions.

Consistent with an increasing risk-free rate, a wealthier optimist implies a lower excess return. All investors agree upon the direction of change, but they disagree regarding levels. The pessimist's perception of excess returns turns negative once the optimist becomes a significant market force. The pessimist sells the stock short, using the proceeds to lend ever more to the optimist, who immediately turns his borrowed money around and invests most of it back in the stock! There is a good deal of trade, but the net effect on investment in the firm is rather slight, consistent with Figure 3. Both agents pursue flawed portfolio strategies corresponding to their flawed beliefs. However the pessimist's strategy seems particularly bad, as he often shorts the stock and lends the proceeds even though the stock does, in reality, offer positive excess returns (e.g., for $\omega \approx 0.5$). Nevertheless, the pessimist is *not* expected to be driven from the market. The resolution to this apparent contradiction is two-fold. First, his lending is not entirely financed through short sales of the stock (he does have positive net wealth), so the expected return on the pessimist's portfolio remains positive under the econometrician's measure. Second, the pessimist consumes at a significantly lower rate than the optimist, except when the optimist dominates the economy. As we see at the rightmost extreme of the plot, excess returns are in fact negative under the true measure when the optimist dominates the economy. Therefore in situations where the pessimist consumes more than the optimist, his portfolio strategy of shorting the stock and buying bonds is expected to pay off.

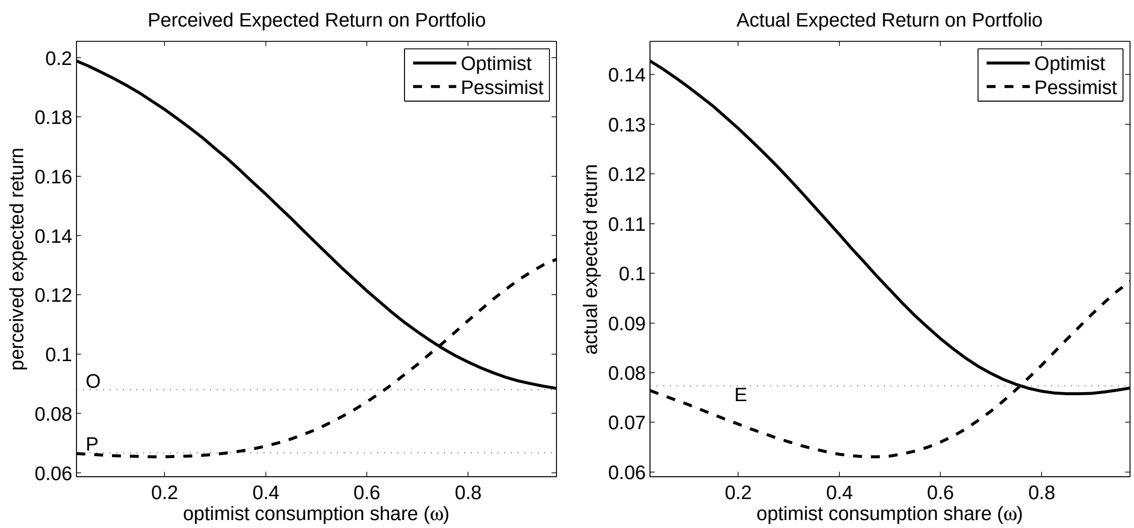

Figure 5: Perceived and actual expected returns on portfolios. In the left panel, the horizontal dotted line marked 'P' indicates the return the pessimist would expect to receive on his portfolio if he were the only agent in the economy (in which case he holds only stock), and likewise the dotted line marked 'O' for the optimist. In the right panel, the dotted horizontal line marked 'E' is the econometrician's homogeneous economy expected portfolio return, which correspond to the true expected return on the stock.

Figure 5 offers a visual summary of the preceding logic, depicting perceived (left panel) and actual (right panel) expected portfolio returns for each agent. Perceived expected portfolio returns are strikingly similar to the plot of individual consumption behavior shown in Figure 2. The horizontal dotted line marked 'P' indicates the return the pessimist would expect to receive on his portfolio if he were the only agent in the economy (in which case he holds only stock). As the optimist becomes wealthier, interest rates rise, and the pessimist begins to buy bonds. When the optimist has more than 40 % of the consumption share, the pessimist thinks that his bond-buying strategy delivers higher expected returns than the underlying technology of the economy (the stock) could generate. For large ω the pessimist views bonds as so undervalued relative to the stock that he is actually more 'optimistic' about expected portfolio returns than the optimist is! This region roughly corresponds to that in which the pessimist consumes more than the optimist. The optimist's perceptions follow roughly a mirror image of the pessimist's: he expects high returns on his levered stock portfolio when the pessimist is dominant. For values greater than $\omega \approx 0.7$, both agents are in effect very optimistic about expected portfolio returns. It is in these situations that consumption booms are observed at the aggregate level.[11]

The actual expected portfolio returns - computed under the econometrician's measure - are of course different. The pessimist's strategy of shorting the stock to buy bonds does not work so well as he thinks. However, his true expected returns are always positive, and when the optimist is dominant the pessimist does actually earn higher expected returns than the optimist. Similarly the optimist's portfolio does offer high expected returns when the pessimist is dominant, although they are not as sensational as he hopes. Recall that these plots do not directly indicate how much of his wealth each agent invests, so higher expected portfolio returns do not imply an expected increase in wealth or consumption share.

4.4 Long-run Impact of Overconsumption

The stock price - equal to the capital stock of the firm - determines the expected rate of economic growth. Whereas the bond market shapes how agents split aggregate output, the stock price influences how much output will be available to split. We observed in Figure 3 that the stock price is typically depressed in the heterogeneous economy, a consequence of overconsumption. However the impact of disagreement upon the stock price is small in percentage terms.

[11] Because agents are risk averse, expected returns are obviously not a sufficient statistic for the desirability of portfolios, although they offer a reasonable summary. For example, when the optimist has a small amount of wealth, the pessimist accepts a lower expected portfolio return than he could achieve by holding the stock, because he trades some expected return for the certainty offered by the bond. For this reason the agents' consumption decisions do not exactly correspond to expected portfolio returns.

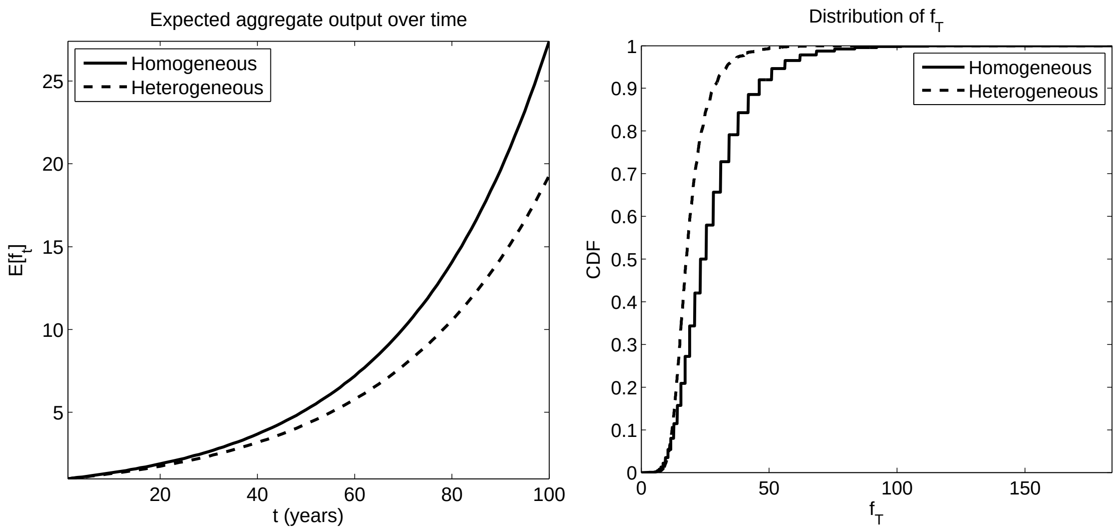

Figure 6: Expected aggregate output over time, and distribution of aggregate output after $T = 100$ years.

Figure 6 assesses the long-run impact of overconsumption on economic growth. In the left panel, the time-path of expected aggregate output in the heterogeneous economy is plotted versus that of a homogeneous economy populated by agents with correct beliefs - those of the econometrician. Recall that the optimist and pessimist split wealth equally at first, that their beliefs average to those of the econometrician, and that they are both expected to survive indefinitely. The slower economic growth in the heterogeneous economy reflects the effects of difference of opinion per se, rather than a mean error in beliefs. Since all investors would choose the econometrician's path of aggregate output if only they had correct beliefs, the lower rate of economic growth may be viewed as a "cost" of disagreement. Although the magnitude of underinvestment is small in each period, compounding amplifies the effect. After a century, aggregate output is expected to be over 30 % higher in the econometrician's economy. The right panel shows that the result carries over to the distribution of aggregate output at the 100 year horizon: based on a Monte Carlo approximation, the econometrician's output stochastically dominates heterogeneous economy output.

5 An Economy with Two Firms

This section presents a model parameterization with two firms, each with access to its own production technology. Whereas the previous section focused on the trade-off between consumption and saving, the objective here is to study the effects of disagreement upon capital allocation *between* firms. Rather than repeating the discussion from the one-firm economy, I focus on a few elements novel to the two-firm setting. The main result is that moderate disagreement regarding one relatively small firm can cause large swings in capital allocations across firms. This occurs without significantly altering consumption/saving decisions at either the individual or the aggregate level.

Agent	Firm 1: $P[H], P[L]$	Firm 2: $P[H], P[L]$
Econometrician	0.55, 0.45	0.45, 0.55
Pessimist	0.55, 0.45	0.42, 0.58
Optimist	0.55, 0.45	0.48, 0.52

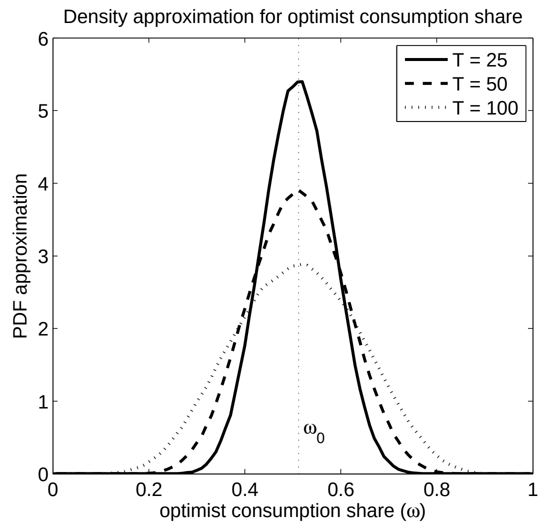

Figure 7: Beliefs and the evolution of the optimist's consumption share. The table at the left shows beliefs regarding the productivity of each firm. Everyone agrees regarding Firm 1's productivity, but there is moderate disagreement regarding Firm 2. The right panel shows a continuous approximation to the probability mass function of ω_t at various horizons.

I maintain all assumptions regarding risk-aversion and initial conditions from the previous section, as given in Table 1, and alter only the specification of productive technologies and agent beliefs. There are two firms, called 1 and 2, each with a access to a single technology. For simplicity the technologies have the same two possible levels of productivity, $\tilde{A}^i \in \{1.2\ (H),\ 0.9\ (L)\}$, $i \in \{1, 2\}$, and I further assume that productivity outcomes are independent across firms. However the technologies differ in the probability that a high or a low productivity state occurs, as detailed in Figure 7. There is complete agreement among agents and the econometrician regarding Firm 1. In addition, all agents perceive Firm 1 as more productive than Firm 2; the productivity of Firm 1 is more likely to be high. However there is disagreement regarding how disadvantaged Firm 2 is. The optimist believes Firm 2 is highly productive with probability 0.48, whereas the pessimist thinks the probability is 0.42. The true probability, known to the econometrician, is 0.45. Because Firm 2 is less productive it will receive less investment in equilibrium than Firm 1, but the benefits of diversification lead to positive investment in each firm.[12] Although I refer to the operators of each technology as "Firms" they are open to interpretation. One reasonable interpretation is that Firm 1 is the agglomeration of most industry, whereas Firm 2 represents a single sector about which there is controversy. Alternatively one could think of the firms as countries, with one more developed than the other.

With the introduction of a second binomial technology process there are now four states of nature ($N = 4$), with states defined over tuples $A \in \mathbb{A} = \{H, L\}^2$. The variables $\tilde{z}$, λ, and ω remain scalars. The right panel of Figure 7 summarizes the dynamics of ω in the 2 Firm economy. Because disagreement between the optimist and the pessimist is less acute in this parameterization, ω_t is more likely to remain close to its origin over long horizons than it was in the previous section.

The expanded state space requires four linearly independent assets to dynamically complete markets. In addition to three natural assets - stock in each firm and one-period bonds - I add an Arrow-Debreu security that pays one unit of the good when both firms experience low productivity. It may be thought of as a sort of "disaster insurance", and has price

$$p_d(\lambda) = E_{\lambda,f}[m(f,\lambda)1_d], \tag{28}$$

where 1_d is an indicator for the disaster state $\{L, L\}$. In practice agents are nearly able to implement their optimal consumption plans without trading disaster insurance - at most it accounts for slightly over 1 % of either agent's portfolio - and so I do not discuss it in subsequent analysis.

The focus of this section is capital allocation, which is presented in Figure 8. The top left panel shows aggregate saving k, the fraction of aggregate output allocated to production rather than consumption. As in the single-firm economy, aggregate saving is highest when the pessimist dominates the economy ($\omega \approx 0$) and lower when the optimist is ascendant ($\omega \approx 1$). However disagreement is more modest in this economy, and the savings curve is now monotonic, unlike the single-firm setting with more severe disagreement. To examine fractional changes in capital allocation, I also show results relative to a fictitious homogeneous economy populated by econometricians, who hold correct beliefs that are an equally weighted average of the

[12]This parameterization is not calibrated to match GDP growth, but expected growth in aggregate output remains reasonable, at 3.25 % per year. To induce positive equilibrium investment in Firm 2 despite its lower productivity it is necessary to increase the variability of the technology somewhat. Therefore the standard deviation of GDP growth is increased to 12.5 %.

optimist and the pessimist. The econometrician's level of saving is the dotted line marked 'E: k', and saving relative to that baseline is shown in the bottom left panel of Figure 8. The range of values for aggregate saving k is very narrow, diverging less than 0.1 % from the econometrician's baseline. By extension there is very little variation in aggregate consumption. This theme extends to individual consumption, where agents over-consume as in the previous section, but by a maximum of 8 basis points relative to wealth (not shown).

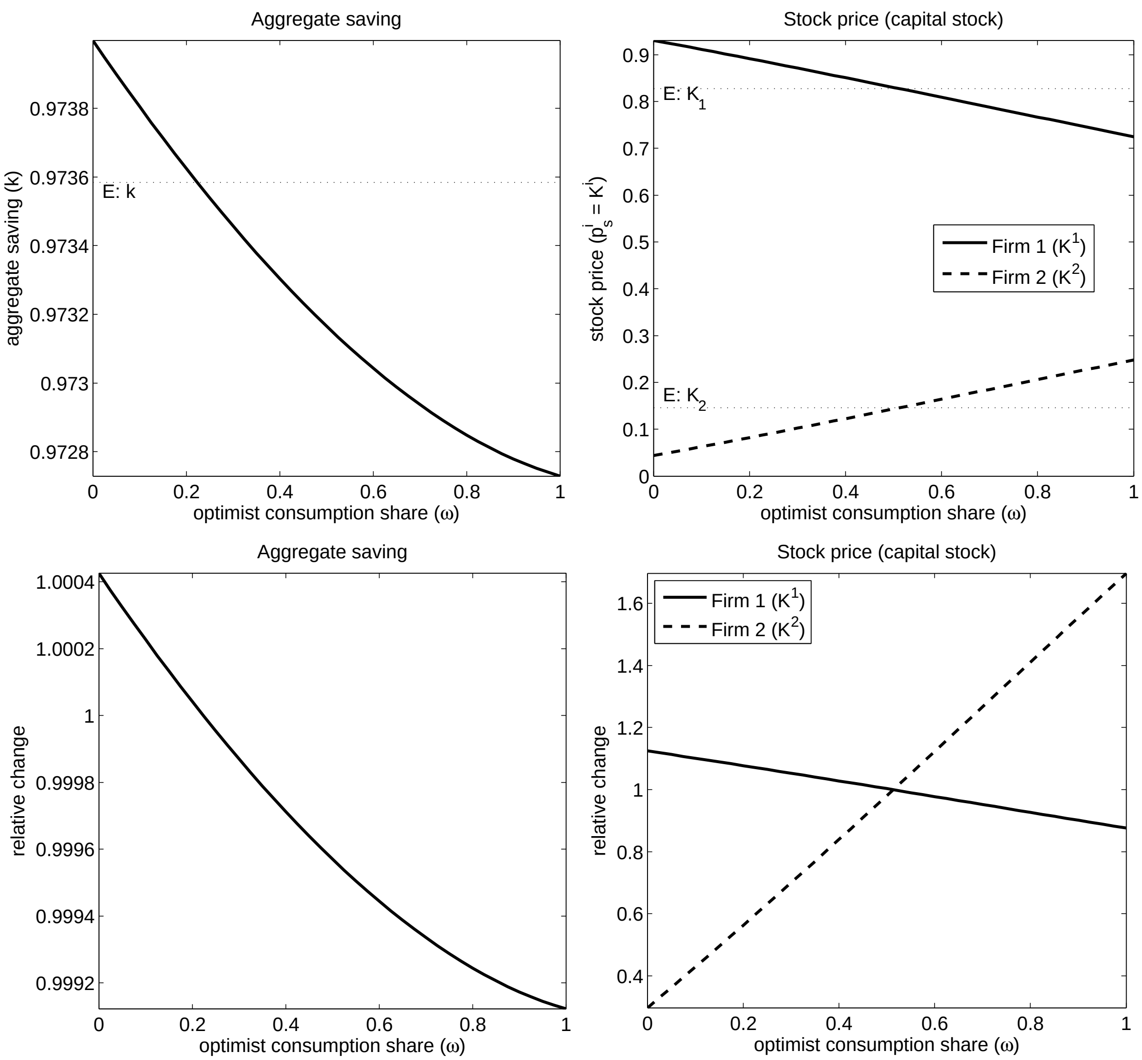

Figure 8: Absolute and relative changes to aggregate and sectoral capital allocation. The dotted lines marked 'E' indicate allocations that would occur in a fictitious homogeneous economy populated by econometricians. The bottom panels are relative to the econometrician's baseline.

This apparent tranquility belies the sensitivity of sectoral allocations to changes in the wealth distribution. The right panels of Figure 8 show capital allocated to each firm, in absolute terms (top) and relative to the econometrician's baseline allocation (bottom). Results in the top panel may also be interpreted as the price of stock in each firm, and I proceed with that interpretation. Horizontal dotted lines give prices in the econometrician's baseline economy. The higher valuation of Firm 1 follows from its higher productivity. Although there is no disagreement regarding Firm 1, "spillover" from disagreement regarding Firm 2 causes investment in the large firm to vary by up to 12 % versus the econometrician's baseline, as illustrated in the bottom right panel. Since aggregate saving remains approximately constant, a decrease in Firm 1's price implies the same *absolute* increase in Firm 2's price. This leads to far larger *relative* changes in the small firm's price. As shown in the bottom right panel, Firm 2 sees variation in excess of 60 % relative to the econometrician's baseline over the domain of optimist consumption share. The caveat is that this change would occur only slowly, as Figure 7 shows that extreme values of ω (near 0 or 1) are unlikely even after 100 years. Annual fractional changes in Firm 2's price, relative to the size of the economy, will only be a few percentage points.

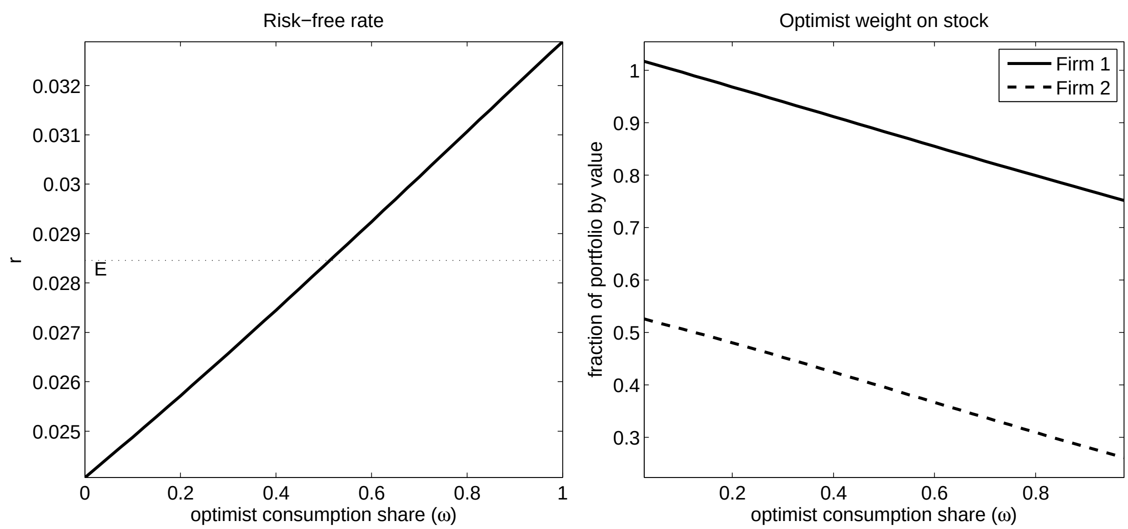

Figure 9: Risk-free rate and the optimist's stock portfolio.

Finally I highlight some aspects of the optimist's portfolio problem. Figure 9 shows the risk-free rate in the left panel, and the optimist's portfolio weights on the two stocks in the right panel. The risk-free rate is roughly 90 basis points lower when the pessimist is dominant than when the optimist is dominant. This is a result of diversification. Because the optimist perceives Firm 2 as only moderately less productive than Firm 1, he invests more in Firm 2 than the pessimist would. The result is a reduction in perceived variability of output, which supports a higher risk-free rate.[13] However, from the standpoint of an optimist choosing his portfolio, the risk-free rate determines the attractiveness of stocks *in general*, as it affects excess returns. The result is that as the optimist's consumption share declines, excess returns increase, and he invests more relative to his wealth in *both* stocks, by borrowing from the pessimist. Individually, the optimist is only directly concerned with the relative weight on Firm 2 in his portfolio, and not with Firm 2's market valuation. The changes in stock prices seen in Figure 8 are mainly due to shifting wealth from the optimist to the pessimist, rather than to changes in the relative weight on each stock in each agent's portfolio.

6 Conclusion

Aggregate consumption may be higher in an economy with optimistic and pessimistic agents than in an economy with only optimistic agents, even though optimists prefer higher consumption than pessimists in homogeneous economies. The result stems from disagreement regarding excess stock returns. The optimist perceives the stock as offering a high excess return, which is driven by the pessimist's willingness to lend at a low risk-free rate. Enticed by "cheap credit", the optimist borrows heavily to finance high current consumption and a large long position in the stock. In contrast, the pessimist expects stock returns to be lower than the risk-free rate, so a strategy of shorting the stock and buying bonds appears very profitable. Each agent believes his portfolio strategy is capable of supporting an elevated level of consumption. As a consequence of overconsumption, economic growth is depressed. I extend my analysis to a multi-sector economy, where controversy regarding a small sector spills over to a large, uncontroversial sector. The altered capital allocation affects the bond market, with implications for agents' individual portfolio strategies.

[13]It is not the case that shifting capital to Firm 2 increases the expected GDP growth. In fact, since all agents agree Firm 1 is the most productive, there is agreement that expected growth is *lower* when following the optimist's policy. The disagreement is over whether too much expected growth is sacrificed to reduce variance.

A Proofs

Proposition 1. *Any function $K(f,\lambda)$ satisfying Equation (11) is homogeneous of degree one in f, and has the form $K(f,\lambda) = fB(\lambda)$, where $B : \mathbb{R}_+ \to \mathbb{R}^M$ satisfies*

$$\left(\frac{1+\lambda^{1/\gamma}}{1-b(\lambda)}\right)^{\gamma} = \beta E\left[\left(\frac{1+(\tilde{z}\lambda)^{1/\gamma}}{1-(\tilde{A}B(\lambda))b(\tilde{z}\lambda)}\right)^{\gamma}\tilde{A}^i\right],\ i \in \{1,\dots,M\} \tag{29}$$

for $b(\lambda) = \sum_{i=1}^{M} B^i(\lambda)$.

Proof. The proof is by contradiction. The proposition only concerns homogeneity w.r.t. f. Since λ evolves independently of f the following applies for general paths of λ_t.

Suppose $\{K_t^*\}$ solves the sequence problem for some f_0^* and λ_0, and let $c_t^* = f_t^* - k_t^*$. Now consider $\hat{f}_0 = \alpha f_0^*$ for constant $\alpha > 0$. Suppose the optimal policy for $\hat{f}_0$ and λ_0 is $\{\hat{K}_t\}$, and that $\{\alpha K_t^*\}$ is not optimal. Clearly the policy αK_t^* defines a feasible consumption plan αc_t^*. Then optimality of $\{\hat{K}_t\}$, $\hat{c}_t = \hat{f}_t - \hat{k}_t$, and homogeneity of $\bar{u}$ in c imply

$$\begin{aligned}
E_0\left[\sum_{t=0}^{\infty}\beta^t\bar{u}(\hat{c}_t,\lambda_t)\right] &> E_0\left[\sum_{t=0}^{\infty}\beta^t\bar{u}(\alpha c_t^*,\lambda_t)\right]\\
E_0\left[\sum_{t=0}^{\infty}\beta^t\bar{u}(\hat{c}_t,\lambda_t)\right] &> \alpha^{1-\gamma}E_0\left[\sum_{t=0}^{\infty}\beta^t\bar{u}(c_t^*,\lambda_t)\right]\\
E_0\left[\sum_{t=0}^{\infty}\beta^t\bar{u}(\frac{\hat{c}_t}{\alpha},\lambda_t)\right] &> E_0\left[\sum_{t=0}^{\infty}\beta^t\bar{u}(c_t^*,\lambda_t)\right]
\end{aligned} \tag{30}$$

But this contradicts the assumption that $\{K_t^*\}$ is optimal for f_t^*, since $\{\frac{\hat{K}_t}{\alpha}\}$ is feasible and supports the superior consumption plan $\{\frac{\hat{c}_t}{\alpha}\}$. Therefore it must be that if $\{K_t^*\}$ solves the problem for f_0^*, then $\{\alpha K_t^*\}$ solves it for αf_0^*. Noting that the recursive formulation of the policy $K(f,\lambda)$ must also solve the sequence problem yields the decomposition $K(f,\lambda) = fB(\lambda)$. Equation (12) derives readily from Equation (11). □

Assumption 1. *Let $\hat{\beta} = \sup_\lambda \beta E\left[\left(\frac{1-b(\tilde{z}\lambda)}{1+(\tilde{z}\lambda)^{1/\gamma}}\right)^{-\gamma}\left(\frac{1-b(\lambda)}{1+\lambda^{1/\gamma}}\right)^{\gamma}(\tilde{A}B(\lambda))^{1-\gamma}\right]$. Assume model parameters s.t. $\hat{\beta} < 1$.*

Proposition 2. *Define a space of continuous, bounded functions $\mathcal{D}(\Lambda)$, $\Lambda \equiv \mathbb{R}_+$, $g \in \mathcal{D}(\Lambda)$ s.t. $g : \Lambda \to [0,1]$, with the sup norm $\|g\| = \sup_{\lambda\in\Lambda}|g(\lambda)|$. Let $\mathcal{T}_D$ be the mapping given by Equation (18),*

$$[\mathcal{T}_D g](\lambda) = \frac{1-b(\lambda)}{1+\lambda^{1/\gamma}} + \beta E\left[\left(\frac{1-b(\tilde{z}\lambda)}{1+(\tilde{z}\lambda)^{1/\gamma}}\right)^{-\gamma}\left(\frac{1-b(\lambda)}{1+\lambda^{1/\gamma}}\right)^{\gamma} g(\tilde{z}\lambda)(\tilde{A}B(\lambda))^{1-\gamma}\right]. \tag{31}$$

Under Assumption 1, there is a unique solution $D(\lambda) \in \mathcal{D}(\Lambda)$ to Equation (18), and $\forall g \in \mathcal{D}(\Lambda)$, $\lim_{N\to\infty}[\mathcal{T}_D^N g](\lambda) \to D(\lambda)$.

Proof. The result follows from the contraction mapping theorem. I first apply Blackwell's sufficient conditions to demonstrate that $\mathcal{T}_D$ is a contraction. Let $g, h \in \mathcal{D}(\Lambda)$ be s.t. $g(\lambda) \leq h(\lambda)$, $\forall \lambda \in \Lambda$. Then $\mathcal{T}_D$ is monotonic, i.e.,

$$[\mathcal{T}_D g](\lambda) \leq [\mathcal{T}_D h](\lambda),\ \forall \lambda \tag{32}$$

since for all realizations z, A of $\tilde{z}, \tilde{A}$ we have

$$\left(\frac{1-b(z\lambda)}{1+(z\lambda)^{1/\gamma}}\right)^{-\gamma} g(z\lambda)(AB(\lambda))^{1-\gamma} \leq \left(\frac{1-b(z\lambda)}{1+(z\lambda)^{1/\gamma}}\right)^{-\gamma} h(z\lambda)(AB(\lambda))^{1-\gamma}, \tag{33}$$

all other terms on either side of the inequality being deterministic and identical. To demonstrate that $\mathcal{T}_D$ has a discounting property, let $g \in \mathcal{D}(\Lambda)$ and $a \geq 0$, a constant. Then

$$\begin{aligned} [\mathcal{T}_D g + a](\lambda) &= \frac{1-b(\lambda)}{1+\lambda^{1/\gamma}} + \beta E\left[\left(\frac{1-b(\tilde{z}\lambda)}{1+(\tilde{z}\lambda)^{1/\gamma}}\right)^{-\gamma}\left(\frac{1-b(\lambda)}{1+\lambda^{1/\gamma}}\right)^{\gamma}(g(\tilde{z}\lambda)+a)(\tilde{A}B(\lambda))^{1-\gamma}\right] \\ &\leq [\mathcal{T}_D g](\lambda) + \hat{\beta} a. \end{aligned} \tag{34}$$

Therefore $\mathcal{T}_D$ is a contraction, and the proposition follows by application of the contraction mapping theorem. □

B Asymptotic Behavior of Optimist Consumption Share (ω)

Survival analysis for the case where agents have identical preferences and impatience but *different magnitudes* of error in their beliefs is presented by Yan [2008]. For the case where errors in beliefs are of *identical magnitude* but where agents nonetheless disagree, e.g., the case of an equally erroneous optimist and pessimist, Yan states that neither agent's consumption share converges to zero. However he does not formally characterize its asymptotic behavior. I do so below for a simple representative case.

The analysis is also novel in that the setting is a discrete-time, discrete state economy. I use the baseline parameters listed in Table 1, although I relax the assumption that $\theta^O = \theta^P = 0.5$. Instead I assume for convenience (but w.l.o.g. for asymptotic analysis) that $\lambda_0 = 1$. I focus purely on the asymptotic behavior of the stochastic weighting factor λ and the consequences for consumption share $\omega(\lambda)$. Thus the analysis here is relevant to either an endowment economy or a production economy: the behavior of λ is determined by disagreement regarding the states of nature, and is independent of whether those states correspond to different realizations of TFP or directly to different levels of aggregate output. The issue is how the pie is split, not its size or origin.

Suppose $\tilde{z}_1 = 3/2$, i.e., a low state L is realized. Then $\lambda_1 = 3/2$, i.e., P's consumption share increases. If $\tilde{z}_2 = 3/2$, another L realization, then $\lambda_2 = (3/2)^2$. A subsequent high state H would bring $\tilde{z}_3 = 2/3$ and $\lambda_3 = 3/2$, moving consumption share back towards O. It should be clear by now that, due to the symmetry of the problem, the state space of λ is

$$\Lambda = \ldots, \left(\frac{3}{2}\right)^{-2}, \left(\frac{3}{2}\right)^{-1}, \left(\frac{3}{2}\right)^{0}, \left(\frac{3}{2}\right)^{1}, \left(\frac{3}{2}\right)^{2}, \ldots \tag{35}$$

We can relabel to the states in Λ according to the exponent i as in $\left(\frac{3}{2}\right)^i$. Further, since realizations of $\tilde{z}$ are determined under the Econometrician's measure, it is equally likely that λ will increase by a factor of 3/2 ($i_t = i_{t-1} + 1$) or decrease by a factor of 2/3 ($i_t = i_{t-1} - 1$). We can model λ by looking at the process for i, which is simply a symmetric random walk on the integers, a well known example of a Markov chain. It has the following properties [14]:

1. It is recurrent, specifically null-recurrent.
2. Consequently any current state $i_t \in \mathbb{Z}$ is revisited with probability 1 as $t \to \infty$, so it cannot be that $\lambda \to 0$ or $\lambda \to \infty$.
3. Null-recurrent Markov chains have no stationary distribution.

Despite the absence of a stationary distribution, we can say something useful about the asymptotic distribution of λ. For $i_0 = 0$ the probability distribution of i_t is approximated by

$$P_t(i) = \begin{cases} \frac{2}{\sqrt{2\pi t}} e^{-i^2/2t}, & t \mod 2 = i \mod 2 \\ 0, & t \mod 2 \neq i \mod 2 \end{cases} \tag{36}$$

[14] See for example Cox and Miller [1980], Hoel et al. [1972].

for values $|i|$ much smaller than t. [15] This formula gives us the probability that i_t takes a value close to the origin for large t. Taking limits, we can see that for any finite distance d from the origin the probability $|i| < d$ is

$$P[|i| < d] = \lim_{t \to \infty} 2 \sum_{i=0}^{d} P_t(i) = 0, \tag{37}$$

since $P_t(i)$ takes on vanishingly small values near the origin for large t and there are only finitely many states within d of the origin. Since this is a symmetric random walk, it is equally likely that i will be either very negative or very positive at long time horizons. It follows that λ_t is asymptotically very nearly 0 or very nearly infinite, each with a roughly 50% chance, but it converges to neither. Consequently consumption share will belong almost entirely to one agent or the other asymptotically, but neither agent is extinguished.

[15] Derivation of this result is shown on http://galileo.phys.virginia.edu/classes/152.mf1i.spring02/RandomWalk.htm. More precise but less elegant results (with printed references) are at http://mathworld.wolfram.com/RandomWalk1-Dimensional.html. These results seem to be pretty well known and are kind of floating around, but I can track down proper sources if this survival discussion is something that should go in the paper.

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