Disagreement, Speculation, and Aggregate Investment*

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Abstract

When investors disagree, speculation between them alters equilibrium prices in financial markets. Because managers maximize firm value given financial market prices, disagreement alters firms' value-maximizing investment policies. Disagreement therefore impacts aggregate investment, consumption, and output. In a production economy with recursive preferences and disasters, we demonstrate that static disagreement among investors generates dynamic aggregate investment that is positively correlated with capital shocks, leading to stochastic volatility in aggregate consumption, investment and equity returns. The direction of these effects is consistent with business cycle facts, and with several features of the 2008 financial crisis.

1 Introduction

It is the summer of 2007 in the United States. High capital investment has accompanied rising equity valuations. An optimistic shareholder, anticipating strong economic growth, increases his risk exposure. His pessimistic counterparty reduces his risk exposure. Then, growth disappoints. Equity plummets, bonds appreciate. For now, the pessimists have won. Although the firm's optimistic shareholders may yet expect the economy to recover, they nevertheless agree to steps by management to restore the firm's value. The firm must attract wealth from those who have it, i.e., from the pessimists with low risk exposure. Management reduces investment and the pessimists approve the change. By the the summer of 2009, aggregate private investment had dropped by 30%, following an even larger decline in the S&P 500 index. In this way, investment policy may follow market sentiment, in the small and in the large. If the optimists are shrewd, whatever their beliefs about growth, then they will assess this source of risk before they invest: that if growth is worse than they expect, investment policy may shift against them also.

Our paper is about that source of risk: that behavioral biases alter fundamentals through speculative trade. In an otherwise standard production economy, we demonstrate that static disagreement among financial market participants introduces new dynamics into aggregate capital investment. We analyze the mechanism in an economy with two types of investor: an optimist and a pessimist. Investors are able to speculate on their beliefs in a complete and frictionless financial market, where the only exogenous sources of risk are shocks to capital growth. As periods of high or low growth are realized, the shifting fortunes of each investor type are reflected in the pricing kernel. The firm responds with a dynamic investment policy that maximizes its value under that pricing kernel. This occurs even though each investor type would, in isolation, prefer a constant investment policy.

The effect is of particular interest because changes in investment policy correlate with capital growth shocks, and so may either amplify or dampen the impact of the shocks on aggregate consumption or investment. The volatility of fundamentals results endogenously through speculative trade in financial assets. For example, dogmatic disagreement can produce an aggregate investment-capital ratio that is positively correlated with capital growth shocks. This increases the volatilities of investment and stock returns, but smooths aggregate consumption.

In our model economy, investors disagree about the expected growth rate in normal times, and about the likelihood of disasters. Although optimists always speculate with pessimists, the degree of speculation varies over time. Because the size of the bets riding on capital growth shocks varies, the magnitude of the investment policy response to a given shock varies also. Investment, consumption and stock return volatilities, which would be constant absent disagreement, become stochastic. Brownian shocks cause smooth variation in wealth shares and consumption during normal times, whereas disasters cause sudden wealth transfers, leading to large and immediate changes in consumption.

Stochastic volatility in consumption is an important source of risk supporting a high and time-varying equity premium, e.g., in the long run risk literature following Bansal and Yaron [2004]. Our model's ability to generate endogenous stochastic volatility from static primitives provides a theoretical scaffold supporting a crucial assumption of many consumption-based asset pricing models.

The ingredients of our theoretical model are orthodox, but novel in combination. We model disagreement as a difference in the perceived average rate of capital accumulation in normal times, and in the perceived arrival rate of disasters. The complete markets equilibrium is the solution to a planner's problem, which corresponds to a competitive market equilibrium where optimists expect higher stock returns than pessimists. All investors in our economy have the same recursive preferences. Finally, investment is subject to a capital adjustment cost. Because of this cost, when the investment rate is high, incremental investment is more expensive so the value of assets in place, or Tobin's q, is high. As a consequence, the stock market is sensitive to changes in the investment rate. Together these assumptions imply that the investment rate, interest rate, and price-dividend ratio are procyclical.

Our model suggests that disagreement influences stock returns through fundamentals, rather than despite them. Disagreement drives returns through its effects on the investment-capital ratio. Cochrane [1991] shows empirically that the investment-capital ratio negatively predicts long horizon excess stock returns. In our model, the investment-capital ratio also negatively predicts the equity premium. Arif and Lee [2014] show empirically that aggregate corporate investments are affected by, and indeed mirror, waves of investor optimism and pessimism, in line with our model.

There has been a steadily growing literature on models with disagreement amongst investors. Basak [2005] shows how to characterize equilibrium asset prices, and Bhamra and Uppal [forthcoming] solve for asset prices with heterogeneous preferences and disagreement. Gallmeyer and Hollifield [2008] study the impact of a short-sales constraint with disagreement, whereas Osambela [forthcoming] studies how asset prices and liquidity are impacted by disagreement given limited commitment. David [2008] shows that disagreement can significantly increase the equity premium with a low level of risk aversion. Dumas et al. [2009] characterize the impact of disagreement on investors' optimal portfolios and asset prices, and Dumas et al. [2014] show that disagreement can help explain several empirical regularities in international finance. Dieckmann and Gallmeyer [2005] and Chen et al. [2012] study the effects of disagreement about disasters on asset prices. In a model with recursive preferences and disagreement, Borovička [2013] shows that a stationary equilibrium exists for the appropriate choice of parameters.

All these papers are set in endowment economies, in which aggregate consumption follows an exogenous process. Our paper builds upon these works, but turns the central question on its head. Rather than asking how disagreement about fundamentals can drive trade and returns in financial markets, we ask how disagreement manifested as trade in financial markets can drive fundamentals.

Detemple and Murthy [1994] study a production economy with disagreement without capital adjustment costs in which all investors have logarithmic utility. The investment-capital ratio, consumption volatility, stock return volatility and Tobin's q are constant and unaffected by disagreement, while the interest rate and the market price of risk are affected. Our model allows for non-logarithmic investors and capital adjustment costs so that investment-capital ratio, consumption volatility, stock return volatility, Tobin's q, the interest rate and the market price of risk are all affected by disagreement.

Sims [2009] studies a two-period economy in which disagreement about inflation influence investors' portfolios and aggregate investment. Much of the intuition from the model in Sims [2009] extends to our infinite horizon economy, in which investors disagree about the growth of capital rather than inflation. Panageas [2005] studies the effects of disagreement of the type studied by Scheinkman and Xiong [2003] on the relation between Tobin's q and investment rates. He shows in an economy with risk-neutral investors facing short-sales constraints that q is related to investment rates. In contrast, our model features risk-averse investors that are allowed to sell short, and we show how investment, Tobin's q, and the risk premium all react to capital shocks in the presence of disagreement. Collin-Dufresne et al. [2014] study the effects of an experiential learning bias in an overlapping generations economy. Heterogeneous learning about output dynamics by different cohorts has significant valuation and investment effects.

Buss et al. [2013] study the effects of different regulations in a production economy with disagreement. They show that Tobin taxes and short-sales constraints can increase stock return volatility, and that leverage constraints can reduce stock return volatility and increase economic growth. Buss et al. [2013] consider a discrete-time, discrete-state economy with a finite horizon. We consider a continuous-time complete markets infinite horizon economy, which allows us to characterize equilibrium in terms of relatively simple expressions. Instead of considering the impacts of regulation, we provide a complete markets benchmark for how the asset prices, risk premia, consumption and investment behave when investors disagree.

Altı and Tetlock [2014] estimate a structural model in which they solve for an individual firm's optimal investment decision in the presence of biased beliefs. They provide empirical evidence that investors have overconfident and trend following beliefs in a partial equilibrium model, in which there is no feedback from the biased beliefs to the dynamics of aggregate consumption or aggregate asset prices. Our general equilibrium formulation allows us to incorporate such feedback.

2 The model

We study a dynamic general-equilibrium production economy in which we allow for disagreement among investors. Our model extends Pindyck and Wang [2013] to include investors' disagreement about the dynamics of capital stock.

The model is set in continuous time with an infinite horizon. Let K_t denote the representative firm's capital stock, I_t the aggregate investment rate, and Y_t the aggregate output rate. The representative firm has a constant returns to scale production technology:

$$Y_t = AK_t, \tag{1}$$

with constant coefficient $A > 0$. Denote the consumption-capital ratio and the investment-capital ratio as

$$c_t \equiv \frac{C_t}{K_t}, \; i_t \equiv \frac{I_t}{K_t}. \tag{2}$$

Using the constant returns to scale assumption, the aggregate resource constraint is

$$C_t + I_t = AK_t \text{ or } c_t + i_t = A. \tag{3}$$

We use j to indicate investor-specific beliefs. Capital accumulation has dynamics given by

$$\frac{dK_t}{K_t} = \frac{\Phi_j\left(I_t, K_t\right)}{K_t}dt + \sigma dW_t^j + (Z-1)dJ_t^j;\ K_0 > 0, \tag{4}$$

where W_t^j is a standard Brownian motion and J_t^j is a jump process with mean arrival rate λ_j. If and when a jump occurs, K falls to ZK; the percentage drop in K is $1-Z$. The random variable Z is uncorrelated with the Brownian and jump processes, and is independently drawn for each jump from the time-invariant probability density function $f(Z)$, with $0 \leq Z \leq 1$.[1]

The function $\Phi_j\left(I_t, K_t\right)$ measures the effectiveness of converting investment goods into installed capital. As in the neoclassical investment literature, e.g. Hayashi [1982], the firm's adjustment cost is homogeneous of degree one in I_t and K_t. Let $\phi_j\left(i_t\right)$ be the increasing, concave, and quadratic function:

$$\phi_j\left(i_t\right) \equiv \frac{\Phi_j\left(I_t, K_t\right)}{K_t} = i_t - \frac{1}{2}\theta i_t^2 - \delta_j, \tag{5}$$

where $\theta > 0$ is the adjustment cost parameter. One interpretation of the parameter δ_j is the expected depreciation rate in normal times, when no jumps occur.

The expected growth rate of capital is

$$\phi_j\left(i_t\right) + \lambda_j(\mathbb{E}\left[Z\right] - 1) = i_t - \frac{1}{2}\theta i_t^2 - \delta_j + \lambda_j(\mathbb{E}\left[Z\right] - 1), \tag{6}$$

where $\lambda_j(\mathbb{E}\left[Z\right] - 1)$ is the expected percentage decline of the capital stock from jumps.

Upon observation of K_t and i_t it is not possible for investors to distinguish whether shifts in capital are driven by the Brownian shock or by the unobserved drift of capital growth. Investors can disagree about the value of δ. Although investors observe discontinuous jumps in capital, they can disagree about the mean arrival rate λ. There are two

[1] All investors agree on the distribution $f(Z)$.

types of investor, $j \in \{a, b\}$, with $\delta_a \leq \delta_b$ and $\lambda_a \leq \lambda_b$, with at least one inequality strict. We refer to type a investors as optimists and type b investors as pessimists because type a investors perceive a lower expected depreciation in normal times and a lower probability of disasters. The two investors are aware of each others's beliefs but they agree to disagree.

From Girsanov's theorem, the change from b's measure to a's measure, which we call η_t, has dynamics

$$\frac{d\eta_t}{\eta_t} = (\lambda_b - \lambda_a)dt + \left(\frac{\delta_b - \delta_a}{\sigma}\right) dW_t^b + \left(\frac{\lambda_a}{\lambda_b} - 1\right) dJ_t^b; \quad \eta_0 = 1, \tag{7}$$

so that for any $T > t$ measurable random variable X we can write

$$\mathbb{E}_t^a [X_T] = \mathbb{E}_t^b \left[\frac{\eta_T}{\eta_t} X_T\right], \tag{8}$$

where $\mathbb{E}_t^j$ denotes investor j's conditional expectation.

The change of measure process η_t shows how type a investors over-estimate or under-estimate the probability of a state relative to type b investors. Because type a investors are optimistic about the mean expected depreciation ($\delta_a \leq \delta_b$), they see positive Brownian shocks to capital as more probable than type b investors. For that reason positive Brownian innovations in capital coincide with positive Brownian innovations in the change of measure η_t.

Similarly, because type a investors are optimistic about the mean arrival rate of jumps ($\lambda_a \leq \lambda_b$), type a investors see a lower likelihood of disasters. For that reason, if and when a jump occurs, η_t falls to $\frac{\lambda_a}{\lambda_b}\eta_t$. The absence of jumps over a period of time is more consistent with the type a investor's beliefs, meaning that η increases deterministically at a rate $\lambda_b - \lambda_a$ in periods in which disasters are not realized.

3 Equilibrium when investors agree

Before presenting the results with disagreement, we summarize the model solution when all investors are of type j, so all of them agree on the value of the parameters δ_j and λ_j, for $j \in \{a, b\}$. The AK production technology and the adjustment cost function imply that investment opportunities are constant so that the aggregate investment-capital ratio and Tobin's q are constant. We use such a simple benchmark to highlight the dynamic effects of disagreement on the economy.

All investors have recursive preferences of the Duffie-Epstein-Zin type with the same relative risk aversion coefficient $1 - \alpha > 0$, the same constant intertemporal elasticity of substitution $\frac{1}{1-\rho} > 0$ and the same subjective discount rate β, with $0 < \beta < 1$.[2] We assume complete markets.[3] Dumas et al. [2000] show that a competitive equilibrium allocation can be obtained from the solution to the planner's optimization problem:

$$\sup_{\{i_t\}} \inf_{\{\nu_t\}} \mathbb{E}_0^j \left[\int_0^\infty e^{\int_0^t -\nu_\tau d\tau} \beta \frac{1}{\alpha} C_t^\alpha \left(\frac{\alpha - \rho \frac{\nu_t}{\beta}}{\alpha - \rho} \right)^{1-\frac{\alpha}{\rho}} dt \right], \tag{9}$$

subject to:

$$\begin{aligned} C_t &= (A - i_t) K_t, \\ \frac{dK_t}{K_t} &= \phi_j (i_t) dt + \sigma dW_t^j + (Z-1) dJ_t^j, \end{aligned}$$

where ν_t is the endogenous discount rate process introduced in Dumas et al. [2000].

The value function is:[4]

$$V_j = \Lambda_j \frac{1}{\alpha} K_t^\alpha, \tag{10}$$

[2]When $\rho = \alpha$, the preferences are of the CRRA type.

[3]Our assumption that capital accumulation is driven by a compound Poisson process in addition to Brownian shocks requires a continuum of assets for markets to be complete. See for example Pindyck and Wang [2013].

[4]Given the homogeneous preferences and the linearly homogeneous capital accumulation process, the value function is homogeneous of degree α in capital.

where Λ_j is a constant. The constant optimal investment-capital ratio i_j is:

$$i_j = \frac{A + \frac{1-\rho}{\theta} - \sqrt{\left(A - \frac{1-\rho}{\theta}\right)^2 + 2\frac{2-\rho}{\theta}\left(\beta - \rho\left[\frac{A}{2-\rho} - \left(\delta_j + \frac{1}{2}\left(1-\alpha\right)\sigma^2 + \lambda_j \frac{1-\mathbb{E}[Z^\alpha]}{\alpha}\right)\right]\right)}}{2-\rho}. \tag{11}$$

From the aggregate resource constraint in Equation (3), the aggregate consumption-capital ratio is also constant: $c_j = A - i_j$.

Since the optimal investment-capital ratio is constant, the equilibrium interest rate, market price of risk, Tobin's q, consumption growth volatility, and stock return volatility are all constant. Naturally there is no leverage or financial trade.

4 Equilibrium when investors disagree

Building on Dumas et al. [2000] and Borovička [2013], the competitive equilibrium with disagreement is obtained from the solution to the planner's problem:[5]

$$\sup_{\{C_{a,t},C_{b,t},i_t\}} \inf_{\{\nu_{a,t},\nu_{b,t}\}} \mathbb{E}_0^b \left[\int_0^\infty \beta \left\{ \eta_t e^{\int_0^t -\nu_{a,\tau} d\tau} \frac{1}{\alpha} C_{a,t}^\alpha \left[\frac{\alpha - \rho \frac{\nu_{a,t}}{\beta}}{\alpha - \rho}\right]^{1-\frac{\alpha}{\rho}} + e^{\int_0^t -\nu_{b,\tau} d\tau} \frac{1}{\alpha} C_{b,t}^\alpha \left[\frac{\alpha - \rho \frac{\nu_{b,t}}{\beta}}{\alpha - \rho}\right]^{1-\frac{\alpha}{\rho}} \right\} dt\right], \tag{12}$$

subject to:

$$\begin{aligned} C_{a,t} + C_{b,t} &= \left(A - i_t\right) K_t, \\ \frac{dK_t}{K_t} &= \phi_b\left(i_t\right) dt + \sigma dW_t^b + (Z-1) dJ_t^b, \\ \frac{d\eta_t}{\eta_t} &= (\lambda_b - \lambda_a) dt + \left(\frac{\delta_b - \delta_a}{\sigma}\right) dW_t^b + \left(\frac{\lambda_a}{\lambda_b} - 1\right) dJ_t^b, \end{aligned}$$

[5]We show in the Appendix that the real investment decisions in the economy can be implemented by a representative firm that chooses the investment plan to maximize the present value of cash flows given the equilibrium state price density.

where we use the change of measure η_t to write the planner's objective function under the pessimist's probability measure, without loss of generality.

There are two dimensions of the optimization problem: the optimal capital allocation between investment and aggregate consumption, and the optimal individual consumption allocation between the two investors. We first consider the solution of the optimal capital allocation problem, obtaining the optimal investment-capital ratio and aggregate consumption-capital ratio.

Define the Pareto share $x_t \in [0,1]$:

$$x_t \equiv \frac{\eta_t e^{\int_0^t -\nu_{a,\tau} d\tau}}{\eta_t e^{\int_0^t -\nu_{a,\tau} d\tau} + e^{\int_0^t -\nu_{b,\tau} d\tau}}, \tag{13}$$

which is driven by the change of measure η_t.[6] We express the equilibrium in terms of x_t.

By the homogeneity of the planner's problem, the value function can be written as

$$V = \left(\eta_t e^{\int_0^t -\nu_{a,\tau} d\tau} + e^{\int_0^t -\nu_{b,\tau} d\tau}\right) H(x_t) \frac{1}{\alpha} K_t^{\alpha}, \tag{14}$$

where H is a function to be determined. We report in the Appendix the associated Hamilton-Jacobi-Bellman equation and the resulting ordinary differential equation and boundary conditions for H. When x_t tends to zero or one, the function H converges to the homogeneous beliefs solution for the optimist and pessimist, respectively. We numerically solve for $H(x_t)$, the optimal investment-capital ratio $i(x_t)$ and the optimal consumption-capital ratio $A - i(x_t)$.

We now turn to the individual optimal consumption allocation for a given level of aggregate consumption. Let $C_t = [A - i(x_t)] K_t$ be aggregate consumption at time t. The first order conditions for individual consumption in the Hamilton-Jacobi-Bellman

[6]Equation (13) shows that x_t is also driven by the endogenous discount rate processes $\nu_{a,t}$ and $\nu_{b,t}$. We show in the Appendix that these endogenous discount rate processes are entirely driven by x_t.

equation lead to the optimal consumption sharing rule:

$$C_{a,t} = \omega(x_t) C_t; \quad C_{b,t} = [1 - \omega(x_t)]\, C_t, \tag{15}$$

where $\omega(x_t)$ is the consumption share of the optimist defined as:

$$\omega(x_t) = \frac{\left(\frac{x_t}{1-x_t}\right)^{\frac{1}{1-\rho}} \left[\frac{H(x_t)+(1-x_t)H'(x_t)}{H(x_t)-x_t H'(x_t)}\right]^{\left(1-\frac{\rho}{\alpha}\right)\frac{1}{1-\rho}}}{1 + \left(\frac{x_t}{1-x_t}\right)^{\frac{1}{1-\rho}} \left[\frac{H(x_t)+(1-x_t)H'(x_t)}{H(x_t)-x_t H'(x_t)}\right]^{\left(1-\frac{\rho}{\alpha}\right)\frac{1}{1-\rho}}}. \tag{16}$$

Interestingly, the consumption share is driven by disagreement not only directly through x_t, but also indirectly through the impact of x_t on recursive preferences, captured by $H(x_t)$.

The dynamics of x_t under the pessimist's beliefs are

$$dx_t = \mu_x^b(x_t) + \sigma_x(x_t)\, dW_t^b + \psi_x(x_t)\, dJ_t^b, \tag{17}$$

where

$$\sigma_x(x_t) = x_t(1-x_t)\left(\frac{\delta_b - \delta_a}{\sigma}\right), \quad \psi_x(x_t) = \frac{\frac{\lambda_a}{\lambda_b}\frac{x_t}{1-x_t}}{1 + \frac{\lambda_a}{\lambda_b}\frac{x_t}{1-x_t}} - x_t, \tag{18}$$

are the sensitivity of x with respect to Brownian shocks and jumps, respectively. Because $\delta_b \geq \delta_a$ and $\lambda_b \geq \lambda_a$, we have $\sigma_x(x_t) \geq 0$ and $\psi_x(x_t) \leq 0$. $\sigma_x(x_t)$ reaches its highest level when $x = \frac{1}{2}$ and $\psi_x(x_t)$ reaches its lowest level when $x = \frac{\lambda_b - \sqrt{\lambda_a \lambda_b}}{\lambda_b - \lambda_a}$.

The dynamics of aggregate investment growth are

$$\frac{dI_t}{I_t} = \mu_I^b(x_t) + \sigma_I(x_t)\, dW_t^b + \psi_I(x_t, Z)\, dJ_t^b, \tag{19}$$

where

$$\sigma_I(x_t) = \sigma + \frac{i'(x_t)}{i(x_t)}\sigma_x(x_t), \quad \psi_I(x_t, Z) = \frac{i(x_t + \psi_x(x_t))}{i(x_t)} Z - 1. \tag{20}$$

From Equation (20), investment growth volatility is stochastic, with dynamics depending on the investment-capital ratio and the sensitivity of x_t to different types of shocks. Brownian shocks and jumps not only affect capital K_t and output Y_t, but they also change the allocation of capital to investment $i(x_t)$. The endogenous allocation of aggregate investment and consumption is a new channel through which disagreement – both δ-disagreement and λ-disagreement – impact the dynamics of investment growth and consumption growth. In addition, at times of disasters, investment growth is affected by the sharp decrease in capital captured by Z.

Market clearing implies that aggregate consumption dynamics are closely linked to aggregate investment dynamics. Aggregate consumption growth follows the process

$$\frac{dC_t}{C_t} = \mu_C^b(x_t) + \sigma_C(x_t)\, dW_t^b + \psi_C(x_t, Z)\, dJ_t^b, \tag{21}$$

where

$$\sigma_C(x_t) = \sigma - \frac{i'(x_t)}{A - i(x_t)}\sigma_x(x_t), \quad \psi_C(x_t, Z) = \frac{A - i(x_t + \psi_x(x_t))}{A - i(x_t)} Z - 1. \tag{22}$$

The dynamics of the pessimist's consumption growth under the pessimist's beliefs are

$$\frac{dC_{b,t}}{C_{b,t}} = \mu_{C_b}^b(x_t) + \sigma_{C_b}(x_t)\, dW_t^b + \psi_{C_b}(x_t, Z)\, dJ_t^b, \tag{23}$$

where

$$\sigma_{C_b}(x_t) = \sigma - \left(\frac{\omega'(x_t)}{1 - \omega(x_t)} + \frac{i'(x_t)}{A - i(x_t)} \right) \sigma_x(x_t), \tag{24}$$

$$\psi_{C_b}(x_t, Z) = \frac{1 - \omega(x_t + \psi_x(x_t))}{1 - \omega(x_t)} \frac{A - i(x_t + \psi_x(x_t))}{A - i(x_t)} Z - 1. \tag{25}$$

Equation (24) shows that the pessimist's consumption growth in the economy with disagreement reacts to Brownian capital shocks through two new channels relative to an economy with agreement. The first channel is the diffusion of the growth of the pessimist's consumption share, which also operates in endowment economies with disagreement: $-\frac{\omega'(x_t)}{1-\omega(x_t)}\sigma_x(x_t)$. The second channel is the effect of the investment-capital ratio on the pessimist's consumption, which is novel to production economies with disagreement. Depending upon the sign of $i'(x_t)$, investor b's consumption risk may be amplified or dampened through this second channel.

Similarly, Equation (25) shows that the sensitivity of the pessimist's consumption growth to disasters is driven by λ-disagreement. Because investors disagree about the mean arrival rate λ, at times of disasters x_t jumps to $x_t + \psi_x(x_t)$. The consumption shares and the investment-capital ratio jump as well, thereby inducing jumps in the pessimist's consumption. In addition, when a disaster occurs the pessimist's consumption is affected by the sharp decrease in capital, captured by Z.

Following Duffie and Epstein [1992], we use the value function to obtain the state price density under the pessimist's beliefs:

$$\xi^b(x_t, K_t) = \beta e^{\int_0^t -\nu_{b,\tau} d\tau} K_t^{\alpha-1}\left([1-\omega(x_t)][A - i(x_t)]\right)^{\rho-1}\left[H(x_t) - x_t H'(x_t)\right]^{1-\frac{\rho}{\alpha}}. \quad (26)$$

The state price density depends on the fundamental risk in K_t and the Pareto weight x_t. The Pareto weight affects the state price density ξ^b through individual consumption shares $\omega(x_t)$ and the investment-capital ratio $i(x_t)$. The state price density is also affected by disagreement due to its impact in recursive preferences through $H(x_t)$.

We characterize asset prices in the economy by analyzing the interest rate, and a stock that pays a dividend stream equal to aggregate consumption C_t. The stock has endogenous price P_t:

$$P_t = \mathbb{E}_t^b\left[\int_t^\infty \frac{\xi_u^b}{\xi_t^b} C_u du\right]. \quad (27)$$

We can also express the stock price as $P_t = q(x_t)K_t$, where Tobin's q is defined as the ratio of the market value of the firm P_t and the book value of the firm K_t:[7]

$$q\left(x_t\right)=\frac{1}{\phi_j'\left(i\left(x_t\right)\right)}=\frac{1}{1-\theta i\left(x_t\right)},\forall j\in\{a,b\}. \tag{28}$$

The dynamics of stock return are

$$\frac{dP_t+C_t}{P_t}=\mu_P^b\left(x_t\right)+\sigma_P\left(x_t\right)dW_t^b+\psi_P\left(x_t,Z\right)dJ_t^b, \tag{29}$$

where

$$\sigma_P\left(x_t\right)=\sigma+\frac{q'\left(x_t\right)}{q\left(x_t\right)}\sigma_x\left(x_t\right),\quad \psi_P\left(x_t,Z\right)=\frac{q\left(x_t+\psi_x\left(x_t\right)\right)}{q\left(x_t\right)}Z-1. \tag{30}$$

Applying Ito's lemma to the equilibrium pricing measure in Equation (26) gives the interest rate and the market prices of risk. The interest rate r_t is:[8]

$$\begin{aligned}r\left(x_t\right)=&\ \nu_b\left(x_t\right)+\left(1-\alpha\right)\phi_b\left(i\left(x_t\right)\right)+\frac{1}{2}\left(1-\alpha\right)\left(2-\alpha\right)\sigma^2 \\ &-\frac{1}{\xi^b\left(x_t,K_t\right)}\frac{\partial\xi^b\left(x_t,K_t\right)}{\partial x_t}\left[\mu_x^b\left(x_t\right)+\left(1-\alpha\right)\sigma\sigma_x\left(x_t\right)\right]+\frac{1}{2}\frac{1}{\xi^b\left(x_t,K_t\right)}\frac{\partial^2\xi^b\left(x_t,K_t\right)}{\partial x_t^2}\sigma_x\left(x_t\right)^2 \\ &-\lambda_b\mathbb{E}_t\left[\frac{\xi^b\left(x_t+\psi_x\left(x_t\right),ZK_t\right)}{\xi^b\left(x_t,K_t\right)}-1\right].\end{aligned} \tag{31}$$

In the first line of Equation (31), the first term is the pessimist's endogenous discount rate process, the second term is the wealth effect associated with expected capital growth in the absence of Brownian shocks and disasters, and the third term is the precautionary savings effect associated with Brownian shocks to capital. By driving the dynamics of x, δ-disagreement and λ-disagreement indirectly impact the dynamics of the first two terms. The second line of Equation (31) incorporates the direct impact of disagreement on the state price density. In particular, the first term captures an additional wealth effect associated with expected growth of x in the absence of Brownian shocks and disasters,

[7]Because δ_j enters $\phi_j(x_t)$ linearly as a constant, $\phi_j'(x_t)$ is the same for all agents $j\in\{a,b\}$.

[8]The solution for the endogenous discount rate processes $\nu_a\left(x_t\right)$ and $\nu_b\left(x_t\right)$ are given explicitly in the Appendix.

and the correlation of x and K. The last term on the second line is the precautionary savings effect associated with Brownian shocks in x. Finally, the term in the third line is the expected increase in the state price density due to disasters, which is driven by both the jump in x and the jump in K.

The market prices of Brownian risk as perceived by the pessimist and optimist are, respectively:

$$\kappa_b(x_t) = (1-\rho)\,\sigma_C(x_t) - (1-\rho)\,\frac{\omega'(x_t)}{1-\omega(x_t)}\sigma_x(x_t) + (\rho-\alpha)\left[\sigma + \frac{1}{\alpha}\frac{x_t H''(x_t)}{H(x_t) - x_t H'(x_t)}\sigma_x(x_t)\right], \tag{32}$$

$$\kappa_a(x_t) = \kappa_b(x_t) + \frac{\delta_b - \delta_a}{\sigma}. \tag{33}$$

The first term in Equation (32) is the market price of Brownian aggregate consumption risk. The market price of aggregate consumption risk changes with x_t because the diffusion of aggregate consumption growth depends endogenously on x_t. The second term is the standard market price of speculative risk, and the third term captures the impact of recursive preferences on the market price of Brownian risk. The relationship between the two investors' perceptions of the market price of Brownian risk remains simple: Equation (33) shows that they are separated by a constant, and the optimist perceives a higher market price of Brownian risk than the pessimist.

The changes in the state price density when a disaster occurs as perceived by the pessimist and optimist are:

$$\zeta_b(x_t, Z) = \frac{\xi^b(x_t + \psi_x(x_t), ZK_t)}{\xi^b(x_t, K_t)} - 1, \tag{34}$$

$$\zeta_a(x_t, Z) = \frac{\lambda_b}{\lambda_a}\zeta_b(x_t, Z) + \frac{\lambda_b - \lambda_a}{\lambda_a}. \tag{35}$$

Accordingly, we define the ex-ante market prices of disaster risk for $j \in \{a, b\}$ as

$$\zeta_j^E(x_t) \equiv \lambda_j \mathbb{E}_t^j[\zeta_j(x_t, Z)]. \tag{36}$$

As with Brownian risk, the optimistic agent a perceives a higher ex-ante market price of disaster risk than does pessimistic agent b. The relationship between ζ_a^E and ζ_b^E depends only on λ-disagreement: $\zeta_a^E = \zeta_b^E + \lambda_b - \lambda_a$.

The equity premium perceived by investor $j \in \{a, b\}$ is:

$$\mu_P^j(x_t) - r(x_t) = \sigma_P(x_t)\kappa_j(x_t) - \lambda_j \mathbb{E}_t^j\left[\psi_P(x_t, Z)\zeta_j(x_t, Z)\right]. \tag{37}$$

The first term in the risk premium is compensation for Brownian risk. By holding the stock, an investor is exposed to its Brownian risk $\sigma_P(x_t)$, and the equilibrium price of this risk is given by $\kappa_j(x_t)$. The second term in the risk premium is compensation for disaster risk. If and when a disaster occurs, the associated change in price is $-\psi_P(x_t, Z)$, and the change in the state price density is $\zeta_j(x_t, Z)$. Because under j's beliefs the mean arrival rate of disasters is λ_j, the compensation for disaster risk is scaled accordingly. Disagreement drives both terms of the equity premium through x.

5 Numerical example

A numerical example illustrates the effect of disagreement on consumption, investment, asset prices and returns, and model dynamics. Pindyck and Wang [2013] identify parameter values to match macroeconomic and financial market moments in a model identical to ours, but with only one agent. We adopt the same parameter values as Pindyck and Wang [2013] for the pessimistic agent b, who is assumed to have correct beliefs. That is, the objective measure in the economy is the same as in Pindyck and Wang [2013]. Agent a is irrationally optimistic regarding both the mean rate of capital depreciation in normal times, and the mean arrival rate of disasters: $\delta_a < \delta_b$ and $\lambda_a < \lambda_b$. Our parameter values are given in Table 1.

5.1 Benchmark economies with agreement

As a benchmark, we compare economies populated by investors with homogeneous beliefs, as described in Section 3. Because most economic moments are constant absent disagreement, summary statistics adequately characterize the benchmark economies. These are reported in Table 2. Expectations are taken under the beliefs of the representative investor for each case. This exercise demonstrates the direct effects of altering beliefs without introducing disagreement.

The economy in the left column of Table 2 is identical to that of Pindyck and Wang [2013].[9] Financial markets have reasonable characteristics, with an equity premium of over 6%, real interest rate of under 1%, and stock return volatility over 14%. In addition the output growth rate and consumption/investment ratios are plausible. However there is some tension between macroeconomic and financial moments: consumption growth volatility is identical to stock return volatility, and at over 14% is not empirically plausible. This tension arises because the consumption rate and Tobin's q are constant. With only capital shocks to drive both consumption growth and stock returns, they must share the same volatility. In the next section we explore disagreement as a mechanism to endogenously decouple consumption growth and stock returns.

The economy in the right column of Table 2 is composed of all irrationally optimistic investors. Here stock return volatility is slightly lower. Beliefs regarding the depreciation rate δ do not affect volatility, but beliefs about the mean arrival rate λ do, so the lower volatility in the optimist's economy stems entirely from the lower mean arrival rate.

The optimist's beliefs about δ and λ do make him bullish about the rate of capi-

[9] We compute volatility as the square root of the instantaneous variance for the differential process, per the formulas given in Appendix A.5. Pindyck and Wang [2013] use a different definition of volatility.

tal accumulation, leading to an expected output growth rate of over 8%. With $\rho > 0$, an investor who is more optimistic about returns will choose to invest more, as the substitution effect dominates the income effect. The higher growth rate under the optimist's beliefs is a composite of two effects: beliefs about capital accumulation for a given investment-capital ratio, and a higher investment-capital ratio. An economy with the optimist's investment-capital ratio of 4.69%, but under the pessimist's beliefs, would have an output growth rate of 2.85%. Finally, the irrationally optimistic investor's high perceived capital growth rate leads to a relatively high real interest rate of 5.59%.

5.2 The effects of disagreement

Disagreement generates interesting dynamics without adding exogenous random processes. All of the statistics presented in Table 2 must be made conditional on the value of the optimist's Pareto share x, or a monotonic transformation thereof. We therefore characterize equilibrium using a series of figures. All expectations are taken under pessimistic agent b's beliefs, which coincides with the objective measure.

Figure 1 characterizes the dynamics of x. The top and middle plots show the diffusion ($\sigma_x(x)$) and jump ($\psi_x(x)$) coefficients of x, as defined in Equations (18).[10] Any negative shock to capital, whether a negative Brownian shock or the occurrence of a disaster, will decrease x. The striking difference between the diffusion and jump sensitivity lies in the magnitude of the coefficients, and how rapidly the state variable may change. Movement due to Brownian shocks will be slow, such that x is unlikely to change by more than a few percentage points in response to Brownian shocks. However in response to a jump x can immediately decline by over 25%. Through jumps it is possible for static disagreement to endogenously generate large and rapid changes in, for example, the investment rate or stock return volatility. Per Equation (18), the sensitivity of x to jumps depends on the ratio of jump intensities λ_a and λ_b, so x may be sensitive to jumps even if they have only

[10] Agents a and b will agree on $\sigma_x(x)$ and $\psi_x(x)$.

a small effect on capital, that is, realizations of Z close to 1. At the bottom of Figure 1 we see the drift $\mu_x(x)$, defined in Equation (A14), which is always positive. x will increase on average in between jump times, when it suddenly declines.

Although we solve the model in terms of the Pareto share x, the consumption share ω is a more easily interpreted state variable that also rescales key economic quantities in a convenient way. Figure 2 plots the consumption share ω versus x in the top left panel, according to Equation (16). Although ω is monotonic in x, it is nonlinear, with ω nearly flat for most of the domain, then increasing rapidly towards $x = 1$.

We use Monte Carlo simulation to approximate the distribution of ω. Specifically, we start the economy with a value of $\omega_0 = \frac{1}{2}$ and simulate the process for 50, 500, and 1000 years. The bottom left panel of Figure 2 shows the estimated probability densities. We see that the optimist's consumption share ω is likely to increase over time, despite the optimist having incorrect beliefs. For example, after 50 years, ω has mean 0.74, and after 500 years it has mean 0.78. However after 1000 years the distribution of ω is little changed from the 500 year distribution. Borovička [2013] shows that in endowment economies with recursive preferences and disagreement, stationarity obtains for a large range of parameters when $\alpha > \rho$. Our preference parameters satisfy this inequality, and therefore it seems reasonable that our choice of parameters leads to a stationary distribution for the consumption share.

Although the optimist will on average have a large share of aggregate consumption at long horizons, the distribution has broad support, with standard deviation of 22% and skewness of -1.41 after 1000 years. As a consequence, a broad range of values for ω will be realized, with a correspondingly wide range of economic outcomes related to variation in the consumption share. For example, there is a 4% chance that the optimist will have less than a 25% consumption share at long horizons, and about 4% probability that the optimist has greater than 99% of the consumption share. Results for the optimist's wealth share, which is plotted versus x in the top right panel of Figure 2 and shown in distribution in the bottom right panel, are similar. Under our assumption of a fixed population mass for each agent type, variation in consumption and wealth share correspond to social mobility - a measure of the evolution of consumption and wealth inequality.

The mechanisms at work in the model are best understood as the result of interaction between optimists and pessimists in a frictionless competitive market. Therefore we begin by describing prices and returns conditional on ω. These are shown in Figure 3; recall that all results in the figure would be constant absent disagreement. All agents agree on prices, summarized in the top row by Tobin's q for the stock and the interest rate r for the riskless bond. Although q is slightly nonmonotonic for high ω, an increase in ω generally implies a higher q and higher interest rate r.

In the second row, disagreement emerges in perceptions of stock return volatility (middle left plot, defined in Equation (A26)). Except for a small region with $\omega > 0.9$, the covariance between q and capital is positive. Hence disagreement raises volatility under either agent's beliefs relative to the benchmark economies with agreement. Agent a's perceptions of stock return volatility also differ substantially from b's, because they disagree about the mean arrival rate of disasters.

Also in the middle row of Figure 3 we see the equity premium. Naturally the optimist perceives a higher equity premium than the pessimist. This results primarily from different perceptions of the market prices of Brownian risk κ and disaster risk ζ^E, which follows from our assumptions $\delta_a < \delta_b$ and $\lambda_a < \lambda_b$.

The results in Figure 3 establish some basic facts about how optimists perceive risk and expected return relative to pessimists. Abstracting from hedging demands, optimists find equity a more attractive investment than do pessimists, because they perceive less volatile returns and a higher equity premium. Equity is a particularly attractive investment when ω is low: the equity premium is countercyclical. Similarly, pessimists find bonds more attractive, and bonds offer the highest return when ω is high.

Informally, optimists find the most attractive investment opportunities with low ω, whereas pessimists find financial markets most attractive with high ω. In a sense then, each agent type would prefer to be in the minority, exploiting the perceived irrationality of the majority. This is reflected in expected returns on wealth, shown for each agent at the top left of Figure 4. With $\rho > 0$, agents consume more and save less when investment opportunities are perceived as poor. Therefore the pessimist decreases his consumption-wealth ratio with ω, whereas the optimist increases his consumption-wealth ratio with ω. There is also a difference in levels, such that the pessimist consumes more out of his wealth than the optimist except when ω is close to 1. The aggregate consumption-wealth ratio reflects individual consumption-wealth ratios weighted by wealth shares. The net effect is that the aggregate consumption-wealth ratio is declining in ω over most of its domain. The exception is $\omega > 0.9$, where both the pessimist and the optimist are consuming less than the optimist consumes when he dominates the economy.

Figure 5 restates the main results for aggregate behavior in terms of the consumption-capital and investment-capital ratios. The procyclicality of investment and the counter-cyclicality of consumption drive a wedge between investment growth volatility and consumption growth volatility. These volatilities are identical under agreement, which is inconsistent with the empirically observed low consumption growth volatility, and high investment growth and stock return volatility. With disagreement, investment growth volatility is higher than consumption growth volatility for most values of ω, because the investment-capital ratio is positively correlated with capital shocks. The consumption-capital ratio is negatively correlated with capital shocks, and aggregate consumption may even increase after the occurrence of a jump. When a jump occurs, a random fraction $1 - Z$ of the capital stock is destroyed. However a jump will also decrease ω, which will usually increase the consumption-capital ratio in equilibrium. The net of these effects can be positive for sufficiently small $1 - Z$, or negative otherwise. The impact of an average jump, $1 - E[Z]$, is to decrease both investment and consumption, per the bottom left panel of Figure 5. Summarizing the impact of all sources of risk, investment growth volatility may reach 18.6% while consumption growth volatility is as low as 13.1%, as compared to identical volatility of 14.4% in the economy with agreement.

We consider ω a measure of sentiment, because it is the consumption share of the irrational optimist. Arif and Lee [2014] provide empirical evidence that aggregate corporate investment measures market-wide investor sentiment, and that periods of high investment predict low equity returns. The top-left panel of Figure 5 establishes that investment increases with ω, and the center-right panel of Figure 3 shows that the equity premium decreases with ω under the rational investor's beliefs, and generally under the optimistic investors beliefs also. Our dynamic general equilibrium model provides a mechanism for the results in Arif and Lee [2014].

5.3 An example based on the 2008 financial crisis

In the introduction, we motivate our study with events from the 2008 financial crisis. Figure 6 shows aggregate data from 2003-2012, incorporating a sequence of high-profile financial failures during the crisis. These failures correspond to sharp drops in GDP, the investment-GDP ratio, the S&P 500 index, and nominal interest rates, and a rise in the VIX. We illustrate similar dynamics in our model using a path of the economy, with a sequence of jumps whose timing is based on the financial crisis. Specifically, we consider a sequence of jump realizations, each of average size, corresponding to the failure of Bear Stearns in March, 2008, and the failures of Lehman Brothers and bailouts of AIG, Freddie Mac, and Fannie Mae in September, 2008. To simplify the illustration, we assume that realized Brownian innovations equal zero.[11]

The optimist speculates on the absence of jumps, which he regards as less likely than the pessimist. In the good times from 2003-2007, the stock realizes high returns, and

[11] We use the parameters from Table 1. In a 10-year period, the expected number of jumps is 7.34, and the expected capital loss per jump is 4.14%. Therefore clustering of the jumps in our example path is unusual, but the number of jumps is not.

the optimist accumulates wealth. The markets increasingly reflect the optimist's beliefs, so the aggregate investment-output ratio, interest rate, and Tobin's q all increase. The pessimist invests in safer assets that pay off in bad times. In the crisis of 2008, the occurrence of jumps is accompanied by a sudden transfer of wealth from the optimist to the pessimist. Accordingly, the investment-output ratio, interest rate, and Tobin's q all drop. Output drops from the destruction of capital, and subsequent expected output growth drops with the investment-capital ratio. The wealth transfer during the crisis also increases speculative risk. Stock return volatility therefore increases after the arrival of each jump.

6 Conclusion

We provide a tractable continuous-time production economy with recursive preferences to study the impact of investors' disagreement on the allocation of output between consumption and investment, as well as on equilibrium asset prices. In production economies in which investment is chosen optimally, a new dimension of risk driven by disagreement emerges. Disagreement affects not only the shares of aggregate consumption among investors with different views, but also the dynamics of aggregate consumption to be shared among those investors.

We model a simple type of disagreement: all the investors are dogmatic and have a fixed bias in their beliefs. Even in such a simple setting, our model leads to stochastic volatility for aggregate consumption and stock returns, and has several interesting implications for the equilibrium allocation of output to consumption and investment, and for asset prices. Disagreement leads to procyclical investment growth, Tobin's q and interest rates. Disagreement also increases stock return volatility, and leads to a countercyclical price of risk.

Our model reproduces several directional relationships between aggregate quantities and asset prices observed during the 2008 financial crisis. Our results suggest that the distribution of wealth and consumption, which fluctuates as a result of financial trade, partially explains the dynamics of aggregate investment and economic growth.

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A Appendix

A.1 Solution with agreement

Dumas et al. [2000] show that the stochastic variational utility approach we use delivers a recursive problem, which can be solved using standard dynamic programming techniques.[12] Let $V\left(K_t, e^{\int_0^t -\nu_\tau d\tau}\right)$ be the value function of the planner in an economy with agreement, where the first argument is capital, and the second argument is the endogenous discount factor. The Hamilton-Bellman-Jacobi equation for the planner's problem in Equation (9) is

$$\begin{aligned} 0 &= \sup_{\{i_t\}} \inf_{\{\nu_t\}} \left\{ e^{\int_0^t -\nu_\tau d\tau} \left\{ \beta \frac{1}{\alpha} K_t^\alpha \left(A - i_t\right)^\alpha \left[\frac{\alpha - \rho \frac{\nu_t}{\beta}}{\alpha - \rho} \right]^{1-\frac{\alpha}{\rho}} - V_2 \nu_t \right\} \right. \\ &\quad \left. + V_1 K_t \phi_j\left(i_t\right) + \frac{1}{2} V_{11} K_t^2 \sigma^2 - \lambda_j \mathbb{E}_t^j \left[V\left(e^{\int_0^t -\nu_\tau d\tau}, K_t\right) - V\left(e^{\int_0^t -\nu_\tau d\tau}, Z K_t\right) \right] \right\}. \end{aligned} \tag{A1}$$

Given the conjectured value function for the representative investor in Equation (10), the solution is the vector of constants $\left(i_j, \nu_j, \Lambda_j\right)$ that satisfy the first order conditions and the Hamilton-Bellman-Jacobi equation (A1). The solution for the optimal investment-capital ratio i_j is given in Equation (11), and the solution for ν_j and Λ_j are

$$\nu_j = \alpha \frac{\beta}{\rho} + \left(1 - \frac{\alpha}{\rho}\right) \left(A - i_j\right) \left(1 - \theta i_j\right), \tag{A2}$$

$$\Lambda_j = \left[\frac{\beta}{\left(A - i_j\right)^{1-\rho} \left(1 - \theta i_j\right)} \right]^{\frac{\alpha}{\rho}}. \tag{A3}$$

A.2 Solution with disagreement

Dumas et al. [2000] show that the stochastic variational utility approach we use delivers a recursive problem, which can be solved using standard dynamic programming techniques. Let $V\left(K_t, \eta_t e^{\int_0^t -\nu_{a,\tau} d\tau}, e^{\int_0^t -\nu_{b,\tau} d\tau}\right)$ be the value function of the planner in an economy with disagreement, where the first argument is capital, the second argument is the product of the change of measure and the endogenous discount factor of investor a, and the third argument is the endogenous discount factor of investor b. The Hamilton-

[12] For an alternative solution technique yielding the same results in the model with agreement see Pindyck and Wang [2013]. The stochastic variational utility approach we use is useful to solve the model with disagreement.

Jacobi-Bellman equation for the planner's problem in Equation (12) is

$$
\begin{aligned}
0 \quad = \quad & \sup_{\{C_{a,t},C_{b,t},i_t\}} \inf_{\{\nu_{a,t},\nu_{b,t}\}} \left\{ \eta_t e^{\int_0^t -\nu_{a,\tau} d\tau} \left\{ \beta \frac{1}{\alpha} C_{a,t}^{\alpha} \left[\frac{\alpha - \rho \frac{\nu_{a,t}}{\beta}}{\alpha - \rho} \right]^{1-\frac{\alpha}{\rho}} \right. \right. \qquad \text{(A4)} \\
& -V_2 \left[\nu_{a,t} - (\lambda_b - \lambda_a) \right] \Big\} + e^{\int_0^t -\nu_{b,\tau} d\tau} \left\{ \beta \frac{1}{\alpha} C_{b,t}^{\alpha} \left[\frac{\alpha - \rho \frac{\nu_{b,t}}{\beta}}{\alpha - \rho} \right]^{1-\frac{\alpha}{\rho}} - V_3 \nu_{b,t} \right\} \\
& + V_1 K_t \phi_b (i_t) + \frac{1}{2} V_{11} K_t^2 \sigma^2 + \frac{1}{2} V_{22} \left(\eta_t e^{\int_0^t -\nu_{a,\tau} d\tau} \right)^2 \left(\frac{\delta_b - \delta_a}{\sigma} \right)^2 \\
& + V_{12} K_t \eta_t e^{\int_0^t -\nu_{a,\tau} d\tau} (\delta_b - \delta_a) \\
& \left. -\lambda_b \mathbb{E}_t^b \left[V \left(K_t, \eta_t e^{\int_0^t -\nu_{a,\tau} d\tau}, e^{\int_0^t -\nu_{b,\tau} d\tau} \right) - V \left(Z K_t, \frac{\lambda_a}{\lambda_b} \eta_t e^{\int_0^t -\nu_{a,\tau} d\tau}, e^{\int_0^t -\nu_{b,\tau} d\tau} \right) \right] \right\}.
\end{aligned}
$$

Using the dynamics of x_t as well as the conjectured value function in Equation (14), the first order conditions with respect to individual consumptions lead to the optimal sharing rule in Equations (15) and (16). Similarly, the first order conditions with respect to the endogenous discount rates imply that

$$
\begin{aligned}
\nu_a (x_t) &= \alpha \frac{\beta}{\rho} + \beta \left(1 - \frac{\alpha}{\rho} \right) \omega (x_t)^{\rho} \left[A - i (x_t) \right]^{\rho} \left[H (x_t) + (1 - x_t) H' (x_t) \right]^{-\frac{\rho}{\alpha}}, \qquad \text{(A5)} \\
\nu_b (x_t) &= \alpha \frac{\beta}{\rho} + \beta \left(1 - \frac{\alpha}{\rho} \right) \left[1 - \omega (x_t) \right]^{\rho} \left[A - i (x_t) \right]^{\rho} \left[H (x_t) - x_t H' (x_t) \right]^{-\frac{\rho}{\alpha}}. \qquad \text{(A6)}
\end{aligned}
$$

Plugging in the first order conditions for individual consumptions and endogenous discount rates, and using the dynamics of x_t as well as the conjectured value function in Equation (14), we obtain that the function $H(x_t)$ and optimal investment $i(x_t)$ are given by the joint solution of:

$$
\begin{aligned}
0 \quad = \quad & \left\{ \frac{1}{\rho} \left[A - i (x_t) \right] \left[1 - \theta i (x_t) \right] - \frac{\beta}{\rho} + \left(\frac{1}{\alpha} (\lambda_b - \lambda_a) + \delta_b - \delta_a \right) x_t + \phi_b (i (x_t)) \right. \qquad \text{(A7)} \\
& \left. - \frac{1}{2} (1 - \alpha) \sigma^2 - \lambda_b \frac{1}{\alpha} \left(1 - \left[1 + \left(\frac{\lambda_a}{\lambda_b} - 1 \right) x_t \right] \frac{H (x_t + \psi_x (x_t))}{H (x_t)} \mathbb{E}_t \left[Z^{\alpha} \right] \right) \right\} H (x_t) \\
& + \left\{ \left(\frac{1}{\alpha} (\lambda_b - \lambda_a) + \delta_b - \delta_a \right) x_t (1 - x_t) \right\} H' (x_t) \\
& + \left\{ \frac{1}{2} \frac{1}{\alpha} \left(\frac{\delta_b - \delta_a}{\sigma} \right)^2 x_t^2 (1 - x_t)^2 \right\} H'' (x_t),
\end{aligned}
$$

and

$$\begin{aligned} & x_t \left[H(x_t) + (1-x_t) H'(x_t)\right]^{1-\frac{\rho}{\alpha}} \omega(x_t)^{\rho} \\ & + (1-x_t) \left[H(x_t) - x_t H'(x_t)\right]^{1-\frac{\rho}{\alpha}} \left([1-\omega(x_t)]\right)^{\rho} \\ = \; & \frac{1}{\beta} \left[A - i(x_t)\right]^{1-\rho} \left[1 - \theta i(x_t)\right] H(x_t), \end{aligned} \tag{A8}$$

corresponding to the HJB and first order condition for investment, respectively.

The boundary conditions for the function H are:

$$\lim_{x \to 0} H(x) = \Lambda_b, \;\; \lim_{x \to 1} H(x) = \Lambda_a, \tag{A9}$$

with Λ_j defined in Equation (A3).

A.3 Investor's individual wealths

Let $X_{b,t} = D_b(x_t) K_t$ be the wealth of the pessimist. Because $X_{b,t}$ is equivalent to a security that pays out a dividend equal to the optimal consumption of the pessimist, $C_{b,t}$, no arbitrage implies that $X_{b,t}$ satisfies

$$\mathbb{E}_t^j \left[d\left(\xi_t^j X_{b,t}\right)\right] + \xi_t^b C_{b,t} dt = 0. \tag{A10}$$

From Ito's lemma, the dynamics of $X_{b,t}$ are

$$\frac{dX_{b,t}}{X_{b,t}} = \mu_{X_b}^j(x_t)\, dt + \sigma_{X_b}(x_t)\, dW_t^j + \psi_{X_b}(x_t)\, dJ_t^j, \tag{A11}$$

where

$$\mu_{X_b}^j(x_t) = \phi_j(i(x_t)) + \frac{D_b'(x_t)}{D_b(x_t)} \mu_x^j(x_t) + \frac{1}{2} \frac{D_b''(x_t)}{D_b(x_t)} \sigma_x(x_t)^2 + \frac{D_b'(x_t)}{D_b(x_t)} \sigma_x(x_t)\, \sigma, \tag{A12}$$

$$\sigma_{X_b}(x_t) = \sigma + \frac{D_b'(x_t)}{D_b(x_t)} \sigma_x(x_t), \qquad \psi_{X_b}(x_t, Z) = Z \frac{D_b(x_t + \psi_x(x_t))}{D_b(x_t)} - 1, \tag{A13}$$

and

$$\mu_x^b(x_t) = x_t(1-x_t) \left[(\nu_{b,t} - \nu_{a,t}) + (\lambda_b - \lambda_a) - x_t \overline{\mu}^2\right], \tag{A14}$$

$$\mu_x^a(x_t) = \mu_x^b(x_t) + \left(\frac{\delta_b - \delta_a}{\sigma}\right) \sigma_x(x_t). \tag{A15}$$

The state price density dynamics are

$$\frac{d\xi^j(x_t, K_t)}{\xi^j(x_t, K_t)} = -r(x_t)\, dt - \lambda_j \mathbb{E}_t^j \left[\zeta_j(Z, x_t)\right] dt - \kappa_j(x_t)\, dW_t^j + \zeta_j(Z, x_t)\, dJ_t^j. \tag{A16}$$

Based on Equations (A11) and (A16), we use Ito's lemma to obtain the dynamics of $\xi_t^j X_{b,t}$, take expectations and plug into Equation (A10), which after simplification leads to the following ODE:

$$
\begin{aligned}
\left[1-\omega\left(x_t\right)\right] \frac{A-i\left(x_t\right)}{D_b\left(x_t\right)} &= r\left(x_t\right)-\mu_{X_b}^j\left(x_t\right)-\lambda_j \mathbb{E}_t^j\left[\psi_{X_b}\left(x_t, Z\right)\right] \\
&\quad +\kappa_j\left(x_t\right) \sigma_{X_b}\left(x_t\right)-\lambda_j \mathbb{E}_t^j\left[\zeta_j\left(Z, x_t\right) \psi_{X_b}\left(x_t, Z\right)\right],
\end{aligned} \tag{A17}
$$

with boundary conditions

$$
\lim_{x \rightarrow 0} D_b(x)=\lim_{x \rightarrow 0} q(x)=\frac{1}{1-\theta i_b}; \quad \lim_{x \rightarrow 1} D_b(x)=0. \tag{A18}
$$

The wealth of a is obtained from market clearing: $X_{a,t}=P_t-X_{b,t}$. The wealth shares of b and a are $\frac{D_b(x_t)}{q_t(x_t)}$ and $1-\frac{D_b(x_t)}{q_t(x_t)}$, respectively. Finally, the perceived excess return of each investor j on his own wealth is

$$
\mu_{X_j}^j\left(x_t\right)-r\left(x_t\right)=\sigma_{X_j}\left(x_t\right) \kappa_j\left(x_t\right)-\lambda_j \mathbb{E}_t^j\left[\psi_{X_j}\left(x_t\right) \zeta_j\left(x_t, Z\right)\right]. \tag{A19}
$$

A.4 Implementing the planner's investment decisions

In equilibrium both investors agree on the firm's value, which is equal to the observed equity price. Therefore, the investment allocation can be implemented through a representative firm that chooses the investment policy to maximize firm value.

Lemma 1 *Given their subjective beliefs and taking prices as given, both investors agree on the investment policy that maximizes firm value.*

Proof. The optimal value maximizing plan for the firm from the perspective of investor $j \in \{a, b\}$ solves

$$
\sup_{\{i_s\}} \mathbb{E}_t^j\left[\int_t^{\infty} \frac{\xi_s^j}{\xi_t^j} K_s\left(A-i_s\right) d s\right], \tag{A20}
$$

$$
\text{s.t. } \frac{d K_t}{K_t}=\phi_j\left(i\left(x_t\right)\right) d t+\sigma d W_t^j+(Z-1) d J_t^j, \tag{A21}
$$

$$
\frac{d \xi^j\left(x_t, K_t\right)}{\xi^j\left(x_t, K_t\right)}=-r\left(x_t\right) d t-\lambda_j \mathbb{E}_t^j\left[\zeta_j\left(Z, x_t\right)\right] d t-\kappa_j\left(x_t\right) d W_t^j+\zeta_j\left(Z, x_t\right) d J_t^j, \tag{A22}
$$

where ξ_t^j is Investor j's state price density. The maximand in the first equation is the observed stock price P_t. Because investors must agree on observed prices, the maximand is identical for any investor j. Equation (A21) is the dynamics of capital under the subjective measure of investor j, and Equation (A22) is the dynamics of the subjective state price density.

Investor-specific state price densities ξ_t^j and the change of measure η_t are related by:

$$\eta_t \propto \frac{\xi_t^b}{\xi_t^a}. \tag{A23}$$

Using this relation, and the dynamics of the change of measure in Equation (7), leads to consistency on the dynamics of the state price density ξ_t^j under the measure of any investor j in Equation (A22).

Because the optimization problem is the same under the eyes of each investor j, both investors must agree on the optimal investment policy that maximizes firm value, given their subjective beliefs and assuming they take equilibrium prices as given. ■

A.5 Instantaneous volatility

Following Pindyck and Wang [2013], we assume Z_t is i.i.d, with power distribution parameter $\gamma > 0$. Omitting the time subscript, the density of Z is

$$p(Z) = \gamma Z^{\gamma-1};\ 0 \leq Z \leq 1. \tag{A24}$$

Accordingly, moments of Z are

$$E[Z^n] = \frac{\gamma}{\gamma+n}. \tag{A25}$$

It then follows that the instantaneous volatility perceived by investor j of stock return, investment growth, and aggregate consumption growth are, respectively

$$Vol_j\left(\frac{dP_t}{P_t}\right) = \sqrt{\sigma_P\left(x_t\right)^2 + \lambda_j\left[\frac{\gamma}{\gamma+2}\left(\frac{q\left(x_t+\psi_x\left(x_t\right)\right)}{q\left(x_t\right)}\right)^2 - 2\frac{\gamma}{\gamma+1}\frac{q\left(x_t+\psi_x\left(x_t\right)\right)}{q\left(x_t\right)} + 1\right]}, \tag{A26}$$

$$Vol_j\left(\frac{dI_t}{I_t}\right) = \sqrt{\sigma_I\left(x_t\right)^2 + \lambda_j\left[\frac{\gamma}{\gamma+2}\left(\frac{i\left(x_t+\psi_x\left(x_t\right)\right)}{i\left(x_t\right)}\right)^2 - 2\frac{\gamma}{\gamma+1}\frac{i\left(x_t+\psi_x\left(x_t\right)\right)}{i\left(x_t\right)} + 1\right]}, \text{ and} \tag{A27}$$

$$Vol_j\left(\frac{dC_t}{C_t}\right) = \sqrt{\sigma_C\left(x_t\right)^2 + \lambda_j\left[\frac{\gamma}{\gamma+2}\left(\frac{A-i\left(x_t+\psi_x\left(x_t\right)\right)}{A-i\left(x_t\right)}\right)^2 - 2\frac{\gamma}{\gamma+1}\frac{A-i\left(x_t+\psi_x\left(x_t\right)\right)}{A-i\left(x_t\right)} + 1\right]}. \tag{A28}$$

	Parameter	Value
Subjective discount rate (%)	β	4.98
Risk aversion	α	-2.07
Time elasticity	ρ	0.333
Volatility of output growth (%)	σ	13.6
Productivity	A	0.113
Adjustment cost	θ	12
Disaster distribution	γ	23.2
Mean arrival rate of jumps (a)	λ_a	0.245
Mean arrival rate of jumps (b)	λ_b	0.734
Depreciation drift (a, %)	δ_a	-6.12
Depreciation drift (b, %)	δ_b	-2.62

Table 1: Parameter values. The table reports the parameter values used in our numerical examples. Agents a and b agree on all parameter values except the mean arrival rate of jumps λ and depreciation drift δ. Agent b's perception of these parameters matches the objective measure.

	Pessimistic (agent b)	Optimistic (agent a)
Stock return volatility (%)	14.414	13.844
Investment growth volatility (%)	14.414	13.844
Consumption growth volatility (%)	14.414	13.844
Market price of Brownian risk	0.415	0.415
Market price of disaster risk	0.112	0.037
Equity premium (%)	6.601	5.953
Interest rate (%)	0.802	5.594
Expected output growth (%)	2.007	8.370
Expected capital loss given jump (%)	4.137	4.137
Investment-capital ratio (%)	2.945	4.456
Tobin's q	1.548	2.154

Table 2: Equilibrium outcomes under agreement. The table reports the equilibrium outcomes in homogeneous beliefs economies in which all investors have either pessimistic or optimistic beliefs. Expectations are taken under the beliefs of the representative investor for each case. Parameter values are reported in Table 1. All equilibrium outcomes reported in the table are constant in the economies with agreement.

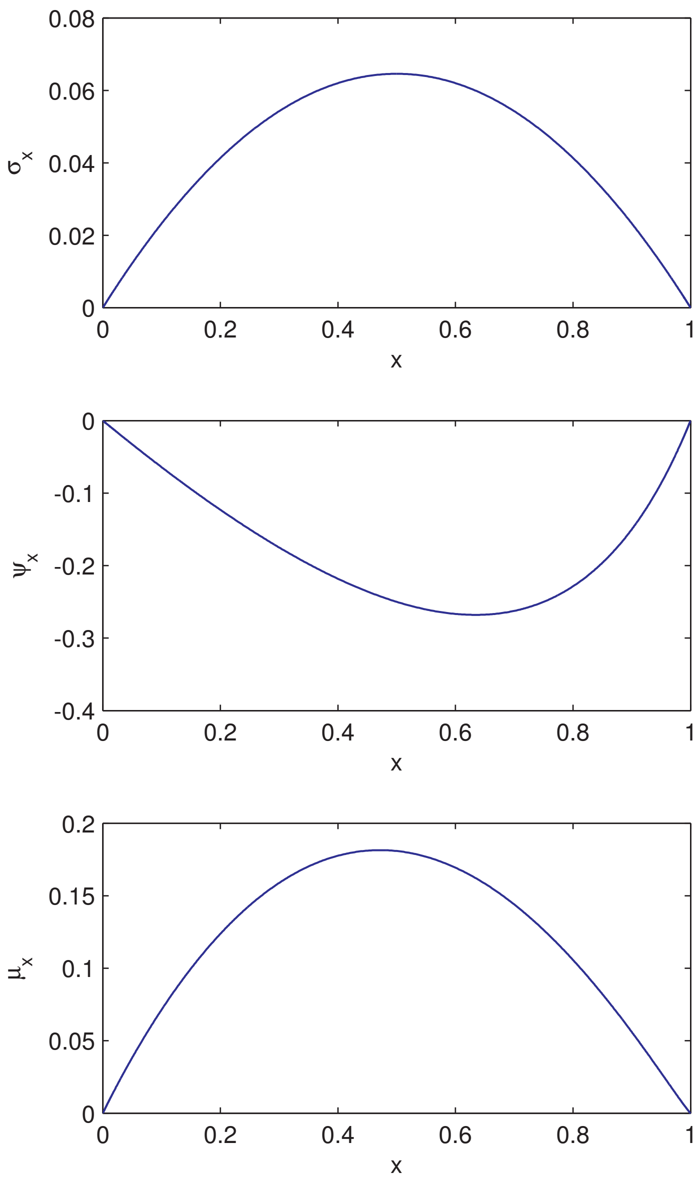

Figure 1: Dynamics of x. The figure shows the diffusion (σ_x, top), jump (ψ_x, middle) and drift (μ_x, bottom) coefficients of the optimist's Pareto share x, conditional on the current value of x. All investors agree on the diffusion and jump coefficients, but they disagree about the drift; the plot shows the drift under the objective measure, corresponding to the beliefs of pessimistic agent b.

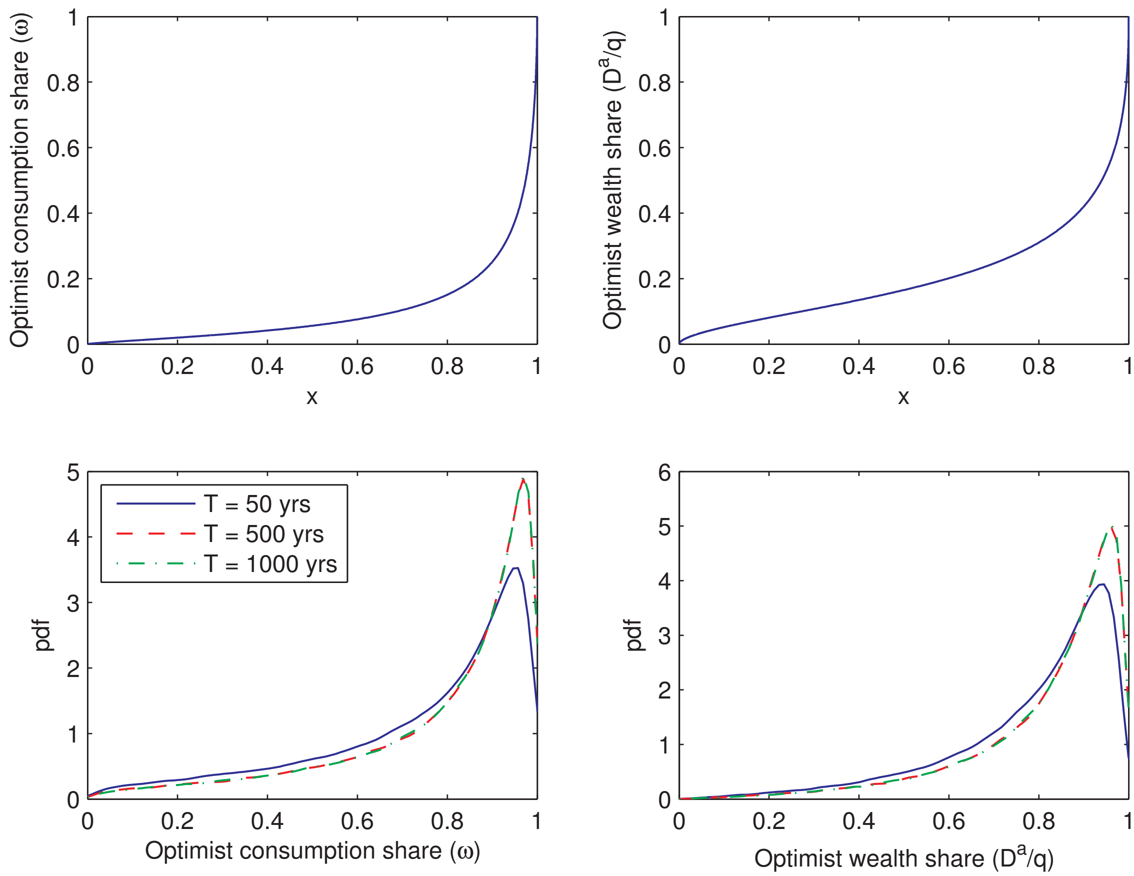

Figure 2: Consumption and wealth shares. The top row maps the Pareto weight x into the optimist's consumption share (ω, left) and the optimist's wealth share (D^a/q, right). The bottom row shows an approximate probability density for ω (left) and D^a/q (right) at 50, 500, and 1000 year horizons, conditional on an initial state corresponding to $\omega_0 = 0.5$. The densities for 500 years and 1000 years are indistinguishable, suggesting a stationary distribution. Densities are approximated using a normal kernel density estimator applied to results from Monte Carlo simulation of the economy with 100,000 paths and time-steps of $dt = 0.1$ year.

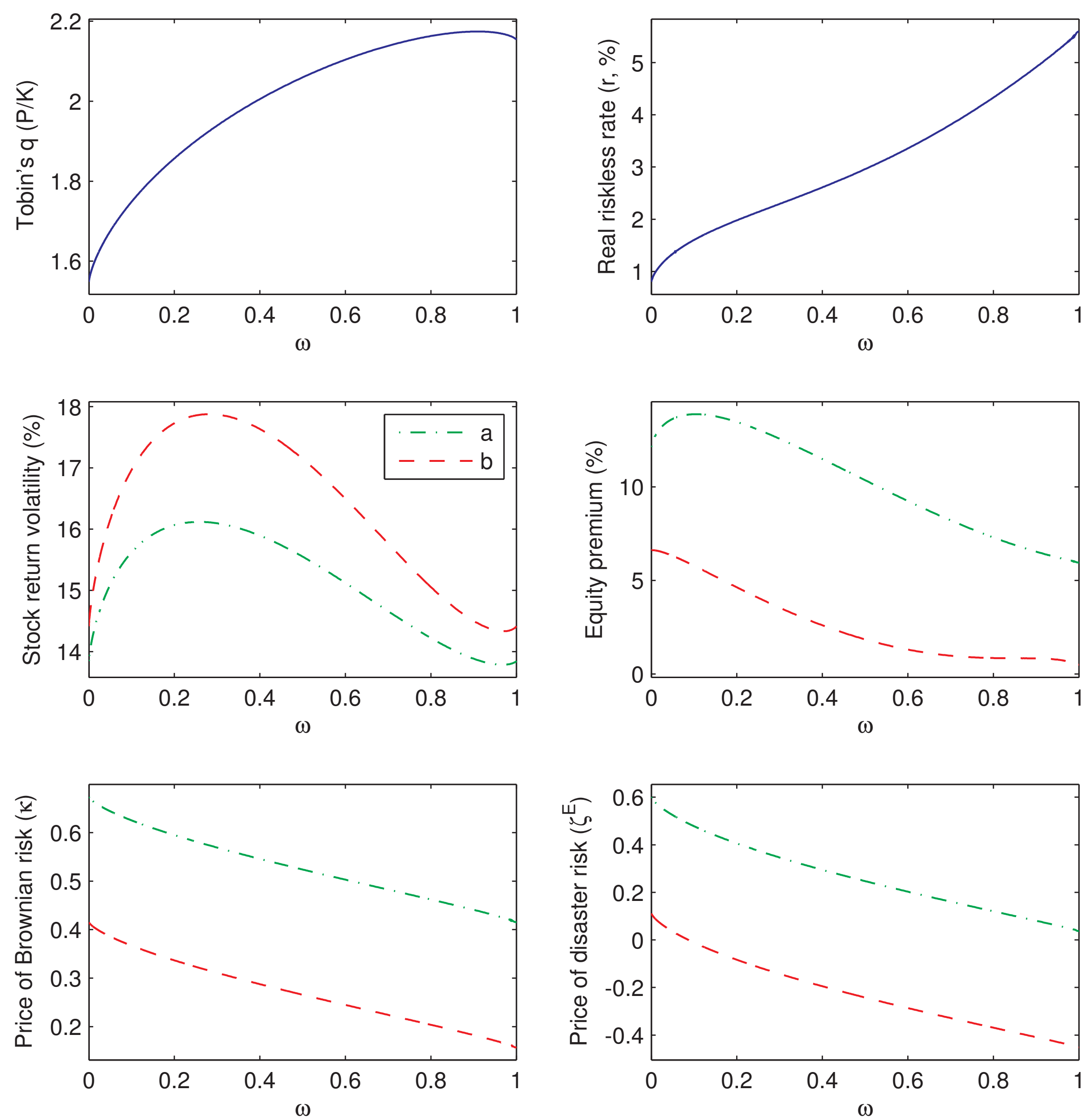

Figure 3: Prices and returns. The figure characterizes financial markets conditional on the optimist's consumption share ω. The top row shows Tobin's q (left) and the real riskless interest rate r (right), upon which all investors agree. Investors may disagree regarding statistics in the bottom two rows of the figure, so results are shown under the perceptions of optimistic investor a (green dot-dashed line) and pessimistic investor b (red dashed line). Agent b's beliefs are correct.

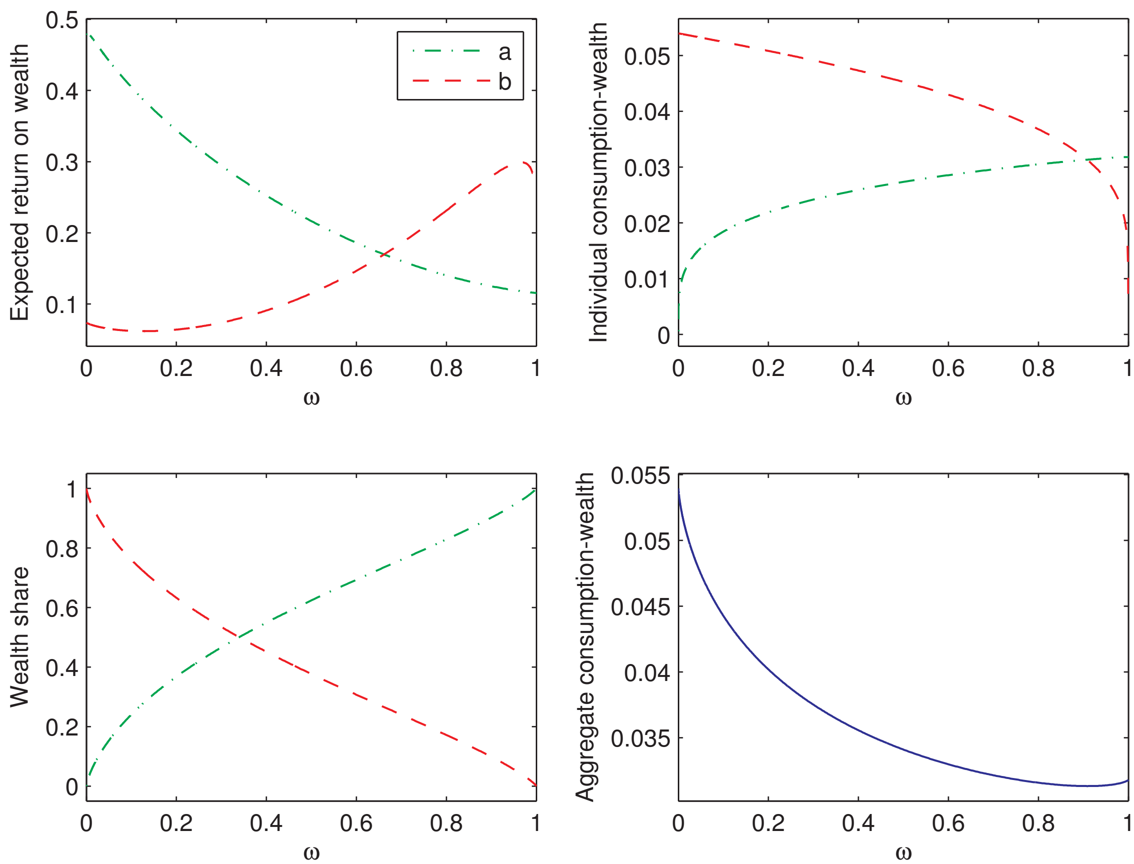

Figure 4: Aggregation of individual consumption decisions. The figure shows how each agent's perception of expected returns on wealth relates to his consumption-wealth ratio (top row), and how the aggregate consumption-wealth ratio reflects a wealth-weighted average of individual decisions. Results are shown for optimistic agent a (green dot-dashed line) and pessimistic agent b (red dashed line) where applicable.

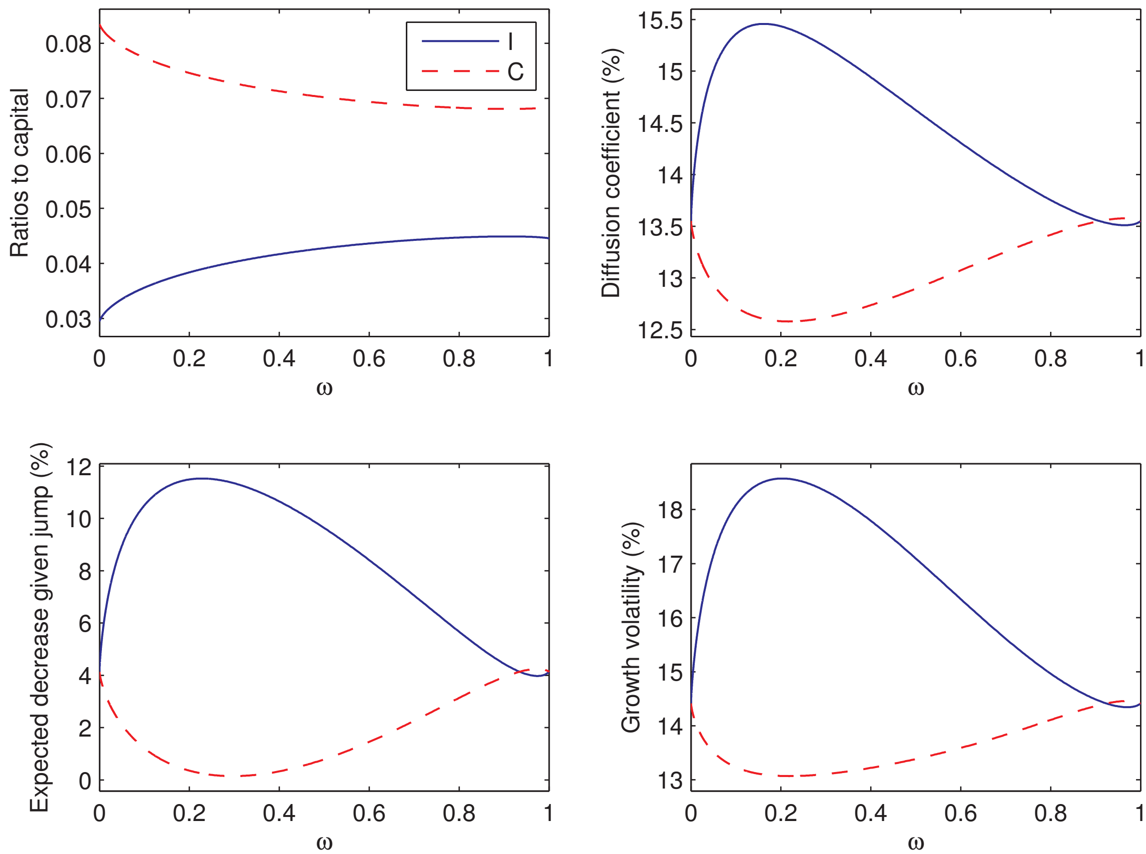

Figure 5: Consumption and investment. The figure shows characteristics of aggregate investment (solid blue line) and consumption (red dashed line) conditional on the optimist's consumption share (ω). In the top left plot, investment and consumption satisfy $\frac{I}{K} + \frac{C}{K} = A$, where K is the capital stock and A is output per unit of capital. The remaining results are expressed as percentages of current consumption or investment, respectively. The expected decrease in investment given a jump occurs is $-E_t\left[\psi_I(x_t, Z)\right]$, and for consumption it is $-E_t\left[\psi_C(x_t, Z)\right]$.

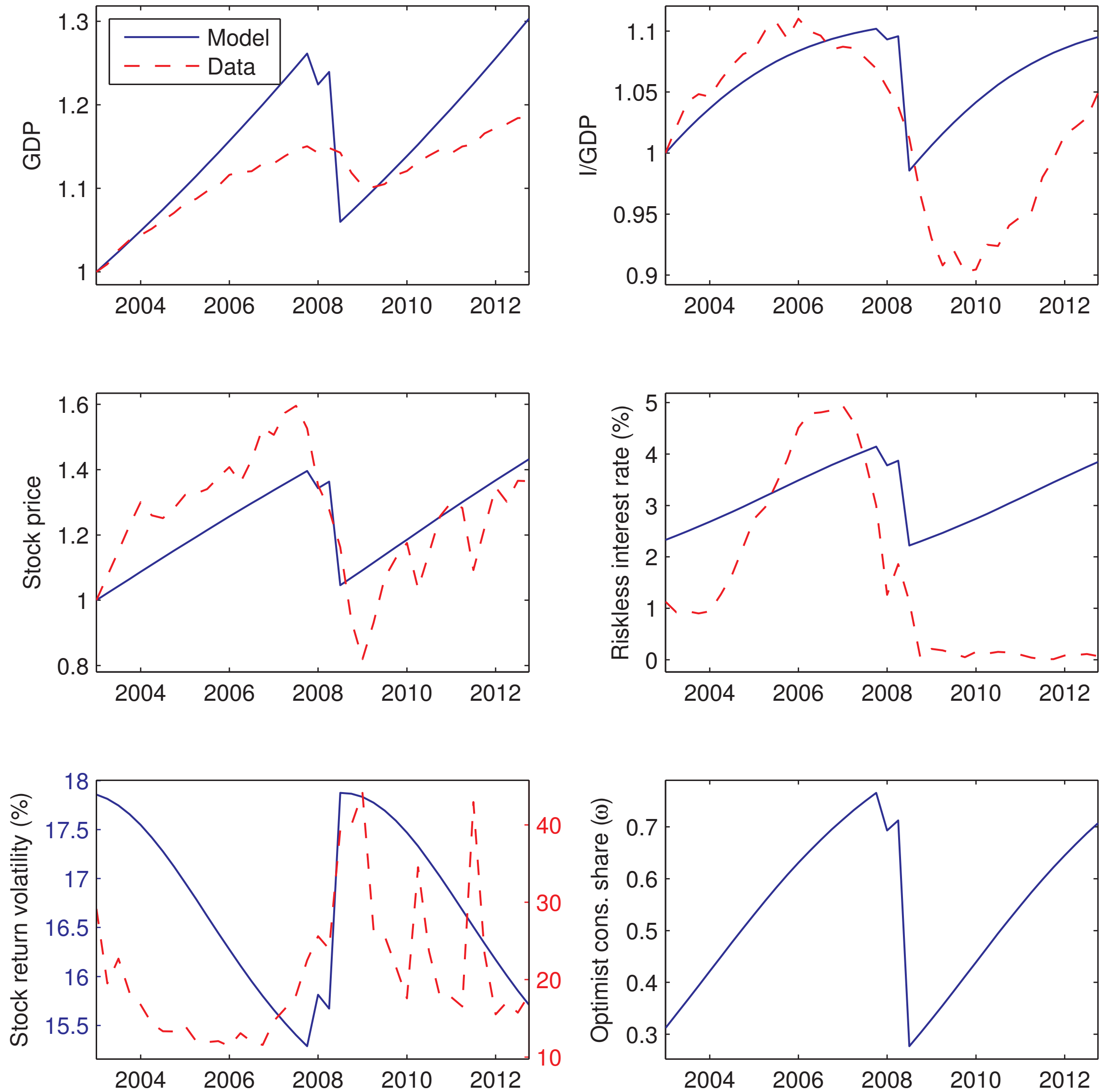

Figure 6: Example model path compared with the 2008 financial crisis. Results from the model (solid blue line) are shown alongside data (red dashed line) from 2003-2012. Data is quarterly, and the model is simulated with $dt = 1/4$ year also. All data is from FRED. US GDP, aggregate investment, and the S&P 500 index (shown as stock price) are expressed in real terms using chained 2009 dollars. GDP is compared to aggregate output Y from the model. Initial values for model and data are normalized to 1. We compare the real instantaneous riskless rate r from the model to nominal 90-day T-Bill yields. The level of the real interest rate is higher in the model path than in the data. Model stock return volatility is shown against the left y-axis (blue), with the VIX plotted against the right y-axis (red). The range of volatility is higher in the data than in the model. The optimist's consumption share (ω) is shown to illustrate changes in the model state.